Further remarks on the collapse of masonry arches with coulomb friction

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ABSTRACT: The purpose of this paper is to analyse the collapse of masonry arches in the presence of finite friction within the framework of the rigid-plasticity theory by adopting an upper bound approach. As usual in dynamics, the decomposition in normal and tangential subproblems is effective to identify the collapse condition by imposing the normality rule for the first one and the principle of maximum dissipation for the second one.

1 INTRODUCTION

The development of the plastic theory during the fifties of the XXth century and Heyman's basic idea of transferring its philosophy from the *steel* to the *stone skeleton* have made it possible to revisit the limit analysis of masonry vaulted systems and to rehabilitate pre-elastic studies on the collapse of the arch.

From then on the limit analysis of the masonry arch in the presence of friction large enough to prevent sliding can be considered a well established problem of linear complementarity characterised by unique solution. As a matter of fact, large values of friction guarantee the validity of the bounding theorems that follows from the normality law between statically admissible stress states and associated flow rule for the displacement field. In this case, the collapse condition can be determined either from below or from above, indifferently.

On the contrary, in the case of sliding due to finite friction the normality law does not hold and the corresponding non-associated flow rule invalidates the bounding theorems. Then, as first observed by Drucker (1954), a gap exists between upper and lower bound and the collapse condition, which is *a priori* not unique, must be investigated within this gap. On the basis of modified bounding criteria (Drucker 1954, Radenkovic 1961, Palmer 1966) computational strategies have been proposed to solve this problem by means of linear programming methods (for instance Livesley 1978, Gilbert and Melbourne 1994, Gilbert et al. 2006). The approach usually adopted has been that of determining the solution starting from the one that dissipates as much energy as friction and obeys a normality law. By this way, however, even though the load multiplier can be sometimes correctly quantified, the corresponding mechanism is surely wrong.

As a matter of fact, the non-associated flow rule requires to tackle the problem in terms of non-linear, non-convex mathematical programming. In this case the search for the optimal solution is difficult not only because of the numerical calculations. The absence of stability criteria makes it possible to find an optimal solution which is not a global minimum, even if the Kuhn-Tucker optimality conditions are verified; moreover, the convergence to the solution strongly depends on the choice of the initial estimate of the unknowns achieved by exploiting the solution of a linear programming problem corresponding to a system with dilatancy rather than friction at the sliding joints (Baggio and Trovalusci 1998, 2000). Thus, in spite of the many sophisticated numerical tools of non-linear analysis now available, appropriate and simple methods for the collapse analysis in the presence of finite friction are still lacking.

The purpose of this paper is to search for the collapse condition as the one that separates equilibrium from starting motion. The arch is treated as a no-tension mono-dimensional continuum with infinite compressive strength in the presence of finite friction, that is a rigid-plastic system subject to friction and unilateral constraints. Once the statics and kinematics of the system have been defined, the kinetics is stated and an upper-bound approach is developed in order to identify possible limit solutions within the domain of the statically admissible states taking into account the principle of maximum dissipation of friction. Thus, by this approach the lower bound is higher than Radenkovic's boundary. As usual in dynamics (Sinopoli 1997), the decomposition of the problem in normal and tangential sub-problems for which collapse conditions and appropriate flow rules can be defined is fundamental in the method proposed. The formulation of the problem is then similar to that of an elastic-plastic system, both for standard and non-standard behaviour. Resultant reactions instead of stress states are considered at the contact joints, with obvious change of sign. Special results have been obtained for a semicircular arch under its own weight in terms of thickness as stability parameter.

2 STATICS

2.1 *Standard behaviour*

Let us consider a semicircular arch of constant thickness with extrados radius R and intrados radius r and let be $K = R/r$. In accordance with the assumed mechanical model, for friction large enough the necessary and sufficient condition for the rotational equilibrium of the system is that a line of thrust can be found wholly lying within the arch thickness. Thus, at the generic joint α with local reference system (***t***, ***n***) as in Fig. 1, the following inequality for the moment ***M*** of the reaction ***R*** with respect to the centre G of the joint must be fulfilled:

$$|M| \leq \frac{h}{2} R^n \tag{1}$$

where h is the thickness of the joint and R^n is the (positive) normal reactive component.

For the symmetry of the system let us analyse a semi-arch and consider the unknown horizontal thrust H applied at the generic point P of the crown joint. Relation (1) satisfied as equality gives the following two boundaries for the thrust H

$$H_{\min}^{r,P}(\alpha) = \frac{Wx_I}{y_I} \tag{2a}$$

$$H_{\max}^{r,P}(\alpha) = \frac{Wx_E}{y_E} \tag{2b}$$

where W is the weight of the voussoir between the crown joint and joint α, and y_I, y_E, x_I, x_E are the arms of the thrust and the weight with respect to the joint intrados I and extrados E.

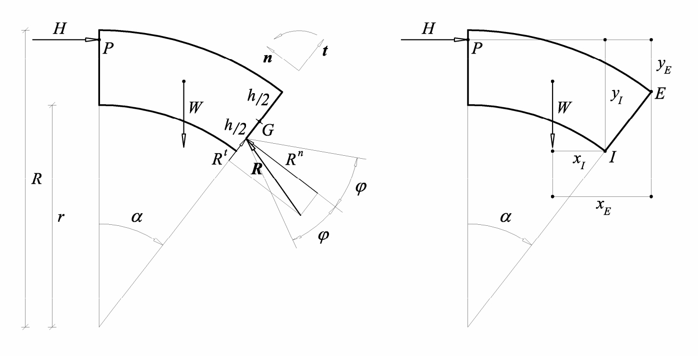


Figure 1 : Scheme of the generic voussoir of the arch.

Analogously, for the translational equilibrium at joint α the following inequality must be fulfilled:

$$\left|R^t\right| \leq \mu R^n \tag{3}$$

where $\mu = \tan\varphi$ and φ are the coefficient and the angle of friction, respectively. Eq. (3) satisfied as equality gives the following two boundaries for the thrust H:

$$H^s_{\min}(\alpha) = \frac{W}{\tan(\alpha + \varphi)} \tag{4a}$$

$$H^s_{\max}(\alpha) = \frac{W}{\tan(\alpha - \varphi)} \quad . \tag{4b}$$

By varying α from the crown joint to the springing and P from the extrados to the intrados of the crown joint the domains H^r and H^s of the statically admissible thrusts can be defined. Obviously, the boundaries of these domains depend on the thickness of the arch, the friction coefficient and the load condition. For a semicircular arch under its own weight with $K = 1.35$ and $\mu = 0.4$ they are shown in Fig. 2, where the domains $H^{r,e}$ and $H^{r,i}$ corresponding to the application point of the thrust at the crown extrados and intrados are also represented. For the assumed value of K the domain H^r is bounded from below by $\max H^{r,e}_{\min}$ corresponding to an 'isostatic' line of thrust with hinges at the crown extrados and the intrados of the joint at 63°, and from above by $\min H^{r,i}_{\max}$ corresponding to an 'isostatic' line of thrust with hinges at the crown intrados and the extrados of the springings. Since the domain H^s must belong to H^r, it is bounded from below by $\max H^s_{\min}$ and from above by $\min H^s_{\max}$ if these extremes belong to H^r, otherwise H^s coincides with H^r and the behaviour is standard.

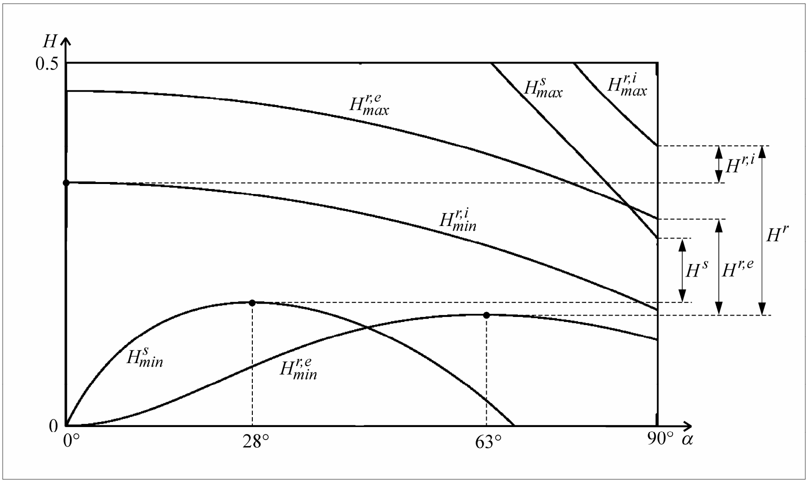


Figure 2 : Domains H^r and H^s for an arch with $K = 1.35$ and $\mu = 0.4$.

2.2 Non-standard behaviour

If H^s does not coincide with H^r as in Fig. 2, the behaviour is non-standard. For $H \in H^s$ the translational equilibrium is always fulfilled, with exception of *H* belonging to the boundary of H^s where the limit condition is attained at a joint. Let *D* be the complementary domain of H^r with respect to H^s . A limit condition of equilibrium is attained for each value of $H \in D$ as it provides a resultant force on the friction cone at a joint (or at most two). Obviously, for the same value of the thrust a range of joints exists where the resultant would go outside the friction cone; in statics, however, this range of joints cannot be associated to the assumed value of *H* (it should be in dynamics), as another thrust $H \in D$ giving a resultant force on the friction cone is associated to each joint where Coulomb's law could be violated. Therefore for each joint a unique value of *H* exists providing that the resultant lies on the friction cone; this correspondence will enable to state a local rule of tangential flow based on the principle of maximum dissipation. Notwithstanding this flow rule, the infinite number of statically admissible limit solutions confirms the *a priori* non-uniqueness of the collapse condition, even though it is not excluded that the solution exists and is unique.

3 KINEMATICS

3.1 Rotational mechanisms

For the feature of the unilateral contact due to the no-tension assumption, the following relation must hold between the relative rotation $\delta\alpha$ at joint α and the relative normal displacement δr^n_G of the centre *G* of the joint:

$$|\delta\alpha| \le \frac{2}{h}\delta r^n_G \quad . \tag{5}$$

Since the formation of rigid-plastic hinges occurs either at the intrados ($\delta\alpha > 0$) or at the extrados ($\delta\alpha < 0$) of the joint, the case of complete separation can be excluded and the local collapse condition implies that the kinematical relation (5) is satisfied as equality. Thus, the possible collapse rotational mechanisms for the whole arch correspond to the fulfilment of (5) as equality at a sufficient number of joints and for appropriate patterns of hinges at the extrados and the intrados. In the case of mechanisms with one degree of freedom there are only two opposite rotational collapse modes. They are represented in Fig. 3 for the most general location of the hinges.

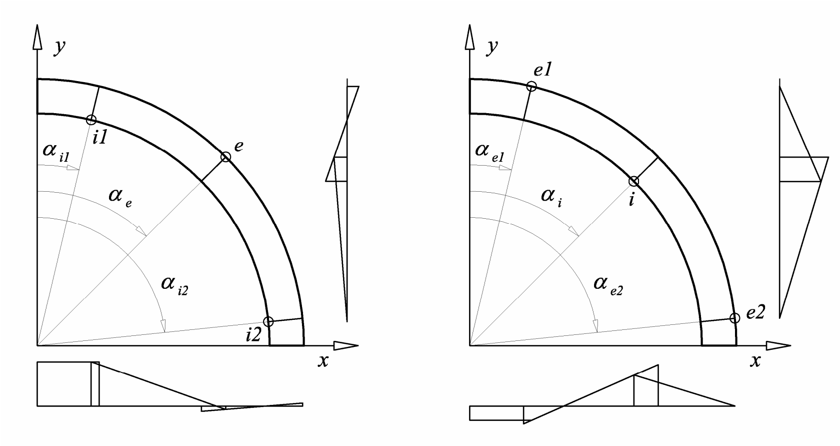
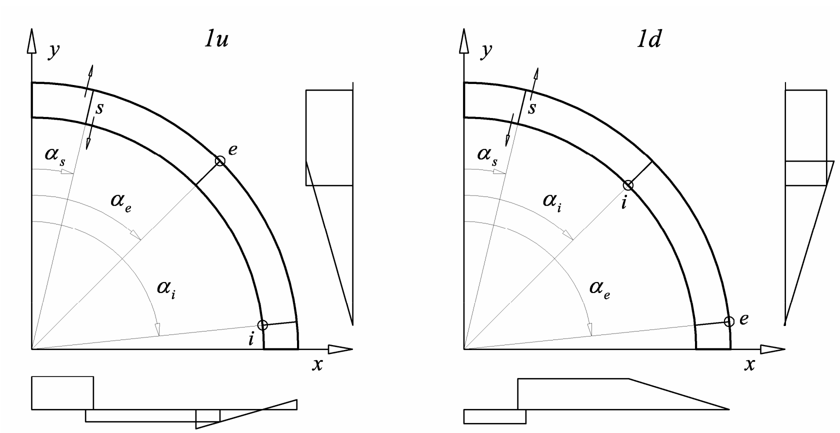


Figure 3 : The two rotational mechanisms with one degree of freedom.

3.2 Mixed and sliding mechanisms

The activation of a relative tangent displacement at joint α depends on the unknown value of the tangential reaction with respect to the limitation imposed by friction. Nevertheless, from a kinematical point of view we can admit the possibility of a virtual sliding δr^t at any joint and later verify in terms of work if it may actually occur for statically admissible values of the thrust corresponding to the resultant force on the friction cone at the same joint. Thus, the possible collapse mechanisms with sliding at a certain joint can be derived by 'coupling' this sliding joint with a compatible choice of rotational joints satisfying (5) as equality. In the case of one degree of freedom these mixed mechanisms are shown in Fig. 4. They also include the case of coincidence of the sliding joint with a rotational joint as in La Hire's mechanism.



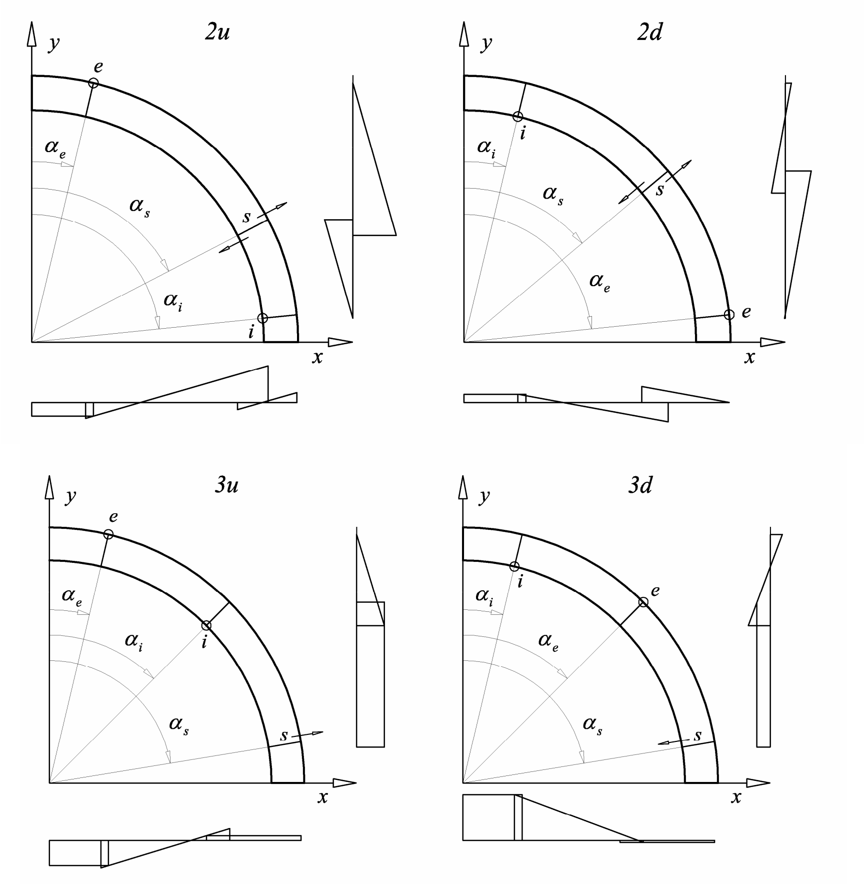


Figure 4 : The six mixed mechanisms with one degree of freedom.

The assumed possibility of activation of sliding at any joint with (5) satisfied as equality also allows sliding mechanisms for the whole arch. They are shown in Fig. 5.

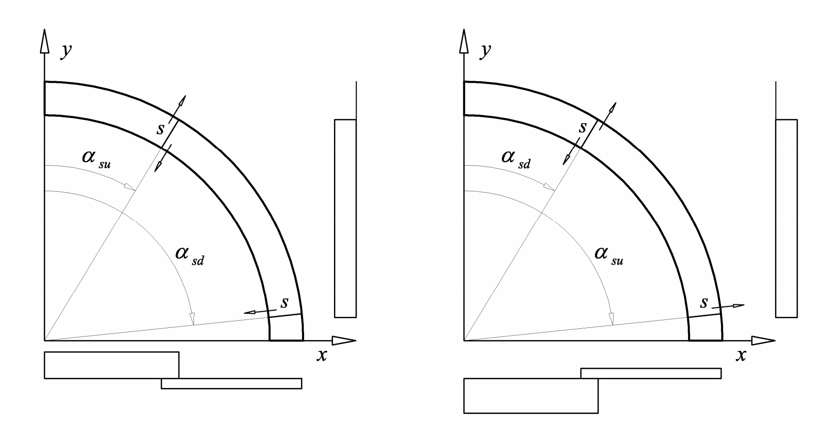


Figure 5 : The two sliding mechanisms with one degree of freedom.

4 KINETICS, EQUILIBRIUM AND LIMIT STATES

4.1 Standard behaviour

The local relations (1) and (5) imply that the statically and kinematically admissible solutions satisfy the following law of normal contact:

$$\delta L^{(r,n)} \geq 0 \tag{6}$$

and that the limit condition corresponds to the normality rule:

$$dL^{(r,n)} = 0 \quad . \tag{7}$$

Eqs. (6) and (7) also hold for the whole arch, for friction large enough to prevent sliding, so that the collapse condition corresponds to a normality rule between static solution and associated collapse mechanism $\boldsymbol{dq}^c$: this means that the ultimate line of thrust touches the extrados and intrados at the same joints where the hinges of the mechanism are located. The search for the limit condition can be formulated as a problem of linear complementarity: the collapse solution is unique and can be found indifferently from above (upper bound) or from below (lower bound). It corresponds to the value K^c for which the domain H^r shrinks to a unique value.

For any admissible mechanism $\delta\boldsymbol{q}$, eqs. (6) and (7) state that for a stable arch:

$$\delta L^{(a)} + \delta L^{(r,n)} = 0 \quad \text{with} \quad \begin{cases} \delta L^{(r,n)} > 0 \quad \text{and} \quad \delta L^{(a)} < 0 \quad \forall\, \delta\boldsymbol{q} \neq \boldsymbol{0} \\ \delta L^{(r,n)} = 0 \quad \text{and} \quad \delta L^{(a)} = 0 \quad \delta\boldsymbol{q} = \boldsymbol{0} \end{cases} \tag{8}$$

where $\delta L^{(a)}$ is the work of the active forces. At the collapse condition it results:

$$dL^{(a)} + dL^{(r,n)} = 0 \quad \text{with} \quad dL^{(a)} = dL^{(r,n)} = 0 \quad \text{and} \quad \boldsymbol{dq}^c \neq \boldsymbol{0} \quad . \tag{9}$$

Thus, the collapse corresponds to the maximum of the potential energy of the active forces.

The local normality rule and the corresponding associate flow become useful tools for coupling local limit solutions - that is ‘isostatic’ lines of thrust touching the extrados and the intrados at certain joints - with the associate pattern of hinges respectful of the normality rule. As it will be shown, these tools will help analyzing the collapse condition for non-standard behaviour as in this case the collapse mechanism results from the ‘composition’ of a proper pattern of rotational and sliding joints.

4.2 Non-standard behaviour

As we have said in *2.2*, if H^r contains H^s the statically admissible values of thrust are those of the domain H^r for which Coulomb’s law is satisfied at each joint and a limit condition of equilibrium is attained for each value of $H \in D$ providing a resultant force on the friction cone. For the principle of maximum dissipation, the tangential reaction R^t makes the virtual work:

$$\delta L^{(r,t)} = R^t \delta r^t \geq -\mu R^n \left|\delta r^t\right| \tag{10}$$

whatever the sign of the tangential displacement may be.

For $H \in D$ let us search now for a collapse mechanism involving sliding with friction. Remind that, at collapse, the local normality rule is valid even though the associate rotational mechanism (without sliding) is null, as it happens for H providing ‘isostatic’ lines of thrust.

This means that $dL^{(r,n)} = 0$ for both mixed and sliding collapse mechanisms. Thus the collapse condition corresponds to the normality rule at the rotational joints and to the principle of maximum dissipation at the sliding joints, that is

$$dL^{(r,n)} = 0 \qquad dL^{(r,t)} = -\mu R^t \left| dr^t \right| \qquad dL^{(a)} = -dL^{(r,t)} > 0 \quad . \tag{11}$$

The maximum dissipation principle assures that, for each collapse mechanism defined by (11), the work of the active forces attains its maximum value with respect to different choices of the sliding joints. In order to find the collapse mechanisms, let us first consider a sliding joint with its associated value of $H \in D$ providing a resultant on the friction cone and then, for the same *H*, let us search for the pattern of hinges satisfying the normality rule at least at two alternate rotational joints. If - for each sliding joint - a 'coupled' pattern of hinges can be found to transform the arch into a mechanism, the corresponding collapse condition is determined.

However, since the friction law does not correspond to a stability criterion, it is expected that no collapse mechanism can be found for any value of the friction coefficient when the domain *D* corresponds to extended ranges of values for *H*. Thus, the collapse mechanisms with sliding must be searched at the intersection of the boundary of H^s and H^r, that is on the boundary of H^r where H^s shrinks to a single value.

5 RESULTS

Let us consider again the semicircular arch with constant thickness under its own weight. For $\mu > 0.395$ and $K > K^c = 1.1136$ the arch is stable. For $\mu = 0.395$ and $K = K^c = 1.1136$ the boundaries of H^r and H^s intersect at a bifurcation point for which coexist both the rotational mechanism and the mixed mechanism with sliding joints at the springings (Fig. 6).

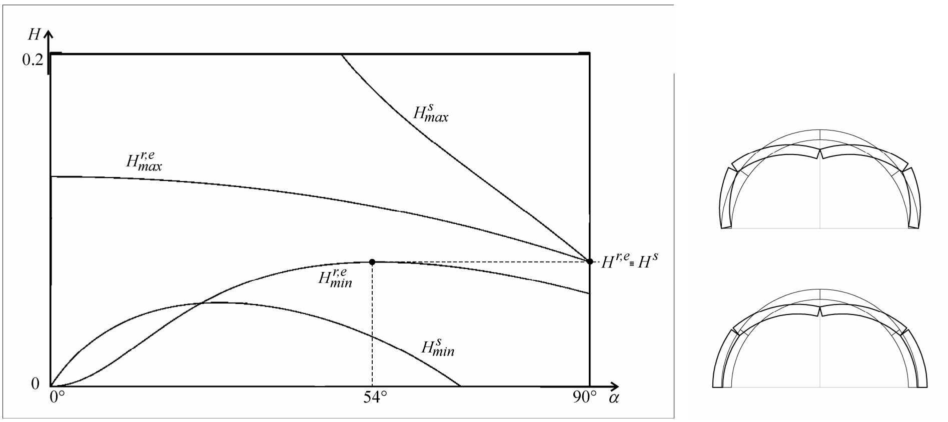


Figure 6 : Collapse condition for both rotational and mixed mechanism ($K = 1.1136$, $\mu = 0.395$).

For $0.309 < \mu < 0.395$ only the mixed mechanism may occur for different intersection points of H^r and H^s corresponding to the range $1.1136 < K < 1.2205$, so that in these ranges of the parameters the uniqueness of the collapse mechanism is guaranteed.

For $\mu = 0.309$ the domain H^s shrinks to a unique value for any *K* (Fig. 7), so that when it intersects H^r (for $K = 1.2205$) another bifurcation point is found for which coexist both the mixed mechanism and the sliding mechanism with joints at 29° and 90°; for $\mu = 0.309$ and $K > 1.2205$ the arch collapses according to the sliding mechanism. Equilibrium is impossible for $\mu < 0.309$.

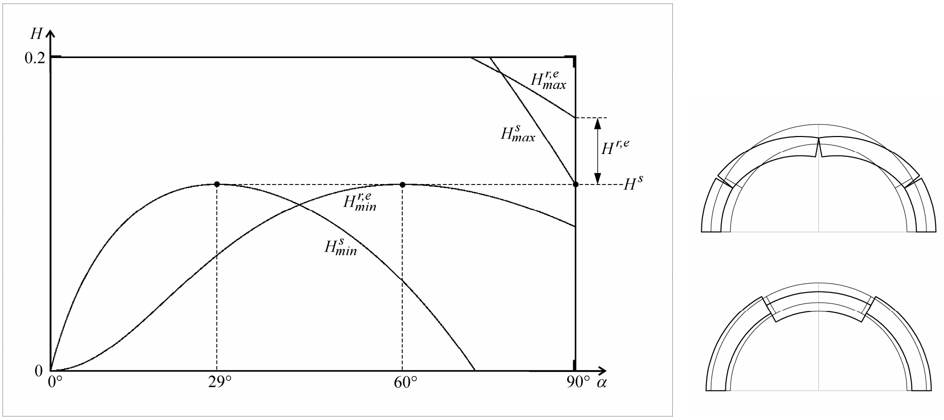


Figure 7 : Collapse condition for both mixed and sliding mechanism ($K = 1.2205$, $\mu = 0.309$).

6 CONCLUSION

The analysis of the collapse mechanisms of the masonry arch in the presence of finite friction has been investigated by means of an upper bound approach by solving the normal and tangent sub-problems. It has been shown that the collapse mechanism is always unique; it defines the collapse thickness of the arch and its typology depends on the value of the friction coefficient. The transition from a collapse mode to another one is characterized by a point of bifurcation.

REFERENCES

Baggio, C. and Trovalusci, P. 1998. Limit analysis for no-tension and frictional three-dimensional discrete systems. *Mechanics of Structures and Machines*, 26, p. 287-304.

Baggio, C. and Trovalusci, P. 2000. Collapse behaviour of three-dimensional brick-block systems using non-linear programming. *Structtural Engineering and Mechanics*, 10(2), p. 181-195.

Drucker, D.C. 1954. Coulomb Friction, Plasticity and Limit Load. *J. of Applied Mechanics*, p. 71-74.

Gilbert, M. and Melbourne, C. 1994. Rigid block analysis of masonry structures. *The Structural Engineer*, 72(21), p. 356-361.

Gilbert, M., Casapulla, C., Ahmed, H.M. 2006. Limit analysis of masonry block structures with non-associative frictional joints using linear programming. *Computer & Structures*, 84, p. 873-887.

Heyman, J. 1966. The stone skeleton. *Int. Journal of Solid and Structures*, 2, p. 249-279.

Heyman, J. 1969. The safety of masonry arches. *Int. Journal of Mechanical Sciences*, 11, p. 363-385.

Livesley, R.K., 1978. Limit analysis of structures formed from rigid blocks, *Int. Journal for Numerical Methods in Engineering*, 12, p. 1853-1871.

Palmer, A. C. 1966. A limit theorem for materials with non-associated flow laws, *Journal de Mécanique*, 5(2), p. 217-222.

Radenkovic, D. 1968. Théorémes limites pour un matériau de Coulomb à dilatation non standardisée, *Comptes Rendus de l'Académie des Sciences*, p. 4103-4104.

Sinopoli, A. 1997. Unilaterality and dry friction. A geometric formulation for two-dimensional rigid body dynamics. *Nonlinear Dynamics*, 12, p. 343-366.

Sinopoli, A., Corradi, M., Foce, F., 1997. A modern formulation for pre-elastic theories on masonry arches. *Journal of Engineering Mechanics*, 2, p. 204-213.