Development of Pippard's elastic method for the assessment of short span masonry arch bridges

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ABSTRACT: The assessment of arch bridge load carrying capacity in UK has traditionally been undertaken using the MEXE (Military Engineering Experimental Establishment) method, which is based on theoretical studies carried out by Pippard. There is a growing concern that the current version of MEXE overestimates the load carrying capacity of short span bridges. This paper presents new equations for working out the safe axle load that might be used for more general application, especially for short span arch bridges where comparison with the results using Pippard's original equations demonstrates the overestimation of the load carrying capacity for short span bridges. The paper also presents the other perceived shortcomings of the Pippard's method.

1 INTRODUCTION

The MEXE method evolved from work undertaken by Pippard in the 1930s which included both field and laboratory tests to calibrate theoretical work. During World War II, this research was used to develop a quick field method to classify bridges according to their capacity to carry military vehicles; this was subsequently adapted for civil use. It has been modified on a number of occasions. The current version of MEXE method is recommended in Departmental Standard BD21 and Advice Note BA16 by the Department of Transport.

Over the years there have several attempts to develop assessment methods for masonry arch bridges. In recent years the MEXE method has been the subject of some criticism in particular with respect to determining the carrying capacity of short span bridges (Havey 2007, McKibbins *et al*. 2006). As part of the UIC 'Masonry Arch Bridges Project' to review the MEXE method of assessment as applied to railway bridges, an investigation was carried out which has allowed several of the assumptions made by Pippard (Pippard 1948) when developing the MEXE method to be revisited and all equations derived from scratch (Melbourne and Wang 2008).

This paper presents further development of Pippard's elastic method for short span arch bridges. New equations are introduced and comparisons have been made with the results using Pippard's original equations (Pippard 1948). It also attempts to identify other limits of Pippard's elastic method.

2 PIPPARD'S ELASTIC METHOD FOR SHORT SPAN ARCH BRIDGES

Pippard used Castigliano's theorems that the partial derivative of the strain energy, U, with respect to a force, is equal to the displacement in the direction of the force. He (Pippard 1948) treated the ring as a two-pin parabolic arch with a secant variation of $I = I_0 \sec \alpha$, as shown in Fig.1, where I_0 is the second moment of area at the crown.
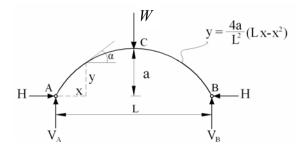
Figure 1 : Pippard's two-pinned parabolic arch ($I = I_0 \sec\alpha$)

Additionally, Pippard ignored the axial thrust and shearing force terms in the strain energy equation. Hence the strain energy was assumed to be totally dependent upon the flexural response of the arch, i.e.

$$U = \int_A^B \frac{M^2 ds}{2EI} \tag{1}$$

where E is modulus of elasticity and ds is increment of length along the arch ring. Thus the value of horizontal reaction at the abutment is given by the solution of the equation

$$\frac{\partial U}{\partial H} = \int_A^B M \frac{\partial M}{\partial H} \frac{ds}{EI} = 0 \tag{2}$$

Total bending moment at x is given by $M = M_s - Hy$, where M_s is the statically determinate bending moment, therefore $\partial M / \partial H = -y$. Substitute the relationships into Eq.(2) gives

$$H = \frac{\int_A^B M_s y \dfrac{ds}{EI}}{\int_A^B y^2 \dfrac{ds}{EI}} \tag{3}$$

Pippard considered a secant variation of the second moment of area, that is, $I = I_0 \sec\alpha$. Therefore, H is given by

$$H = \frac{\int_A^B M_s y \dfrac{ds}{EI}}{\int_A^B y^2 \dfrac{ds}{EI}} = \frac{\int_A^B M_s y \dfrac{dx \sec\alpha}{EI_0 \sec\alpha}}{\int_A^B y^2 \dfrac{dx \sec\alpha}{EI_0 \sec\alpha}} = \frac{\int_A^B M_s y \, dx}{\int_A^B y^2 \, dx} \tag{4}$$

where dx is increment of length along span L
   However, for a short span arch with relatively a thicker ring, the axial thrust term in the strain energy U should be considered, i.e.

$$\frac{\partial U}{\partial H} = \int_A^B \frac{M \dfrac{\partial M}{\partial H}}{EI} ds + \int_A^B \frac{N \dfrac{\partial N}{\partial H}}{EA} ds = 0 \tag{5}$$

where A is the cross section area of the arch ring at any point x. The axial force N at any point x is given by

$$N = H\cos\alpha, \quad \partial N / \partial H = \cos\alpha \tag{6}$$

Substituted Eq.(6) into Eq.(5) gives

$$\int_A^B (M_S - Hy)(-y)\frac{ds}{EI} + \int_A^B (H\cos\alpha)\cos\alpha\frac{ds}{EA} = 0 \tag{7}$$

Therefore, the horizontal reaction at the abutment becomes

$$H = \frac{\displaystyle\int_A^B M_S y \frac{ds}{EI}}{\displaystyle\int_A^B y^2 \frac{ds}{EI} + \int_A^B \cos^2\alpha \frac{ds}{EA}} \tag{8}$$

For the parabolic arch with a secant variation $I = I_0 \sec\alpha$, H becomes

$$H = \frac{\displaystyle\int_A^B M_s y \frac{dx\sec\alpha}{EI_0\sec\alpha}}{\displaystyle\int_A^B y^2 \frac{dx\sec\alpha}{EI_0\sec\alpha} + \int_A^B \cos^2\alpha \frac{dx\sec\alpha}{EA}} = \frac{\dfrac{\displaystyle\int_A^B M_s y dx}{\displaystyle\int_A^B y^2 dx}}{1+\lambda} \tag{9}$$

Where

$$\lambda = \frac{\displaystyle\int_A^B \cos\alpha\frac{I_0}{A}dx}{\displaystyle\int_A^B y^2 dx} = \frac{5}{32}\left(\frac{d}{a}\right)^2 \int_0^1 (\cos\alpha)^{\frac{4}{3}}dk \tag{10}$$

where dk is increment of length along unit span.

When the span/rise ratio is fixed, the integration of $(\cos\alpha)^{\frac{4}{3}}$ is a constant, so λ is in direct proportion to $(d/a)^2$.

Fig.2 shows the changes of λ for different ring thickness and span rise ratio.
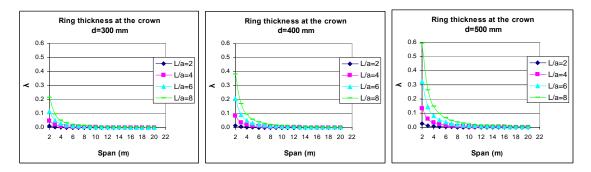

Figure 2 : Value of λ for different span rise ratio and ring thickness

It can be seen from Fig.2 that for the bridges of the same ring thickness, the influence of considering the axial thrust term becomes significant for relatively small span bridges. As the span increases, λ gradually drops to zero, which means that there is little difference between the results whether including or neglecting the axial thrust term in the strain energy, i.e. Eq.(4) and (8) produce the same horizontal reaction at the abutment. With the increase of the ring thickness, the value of λ increases correspondingly, indicating that the influence of considering the axial thrust term becomes more significant for relatively thicker arch rings.

Table 1 shows the comparison of equations for live load effects when neglecting the axial thrust term in working out the strain energy and when incorporating the effects of axial strain energy whilst Table 2 shows the comparison of equations for dead load effects.

A full derivation of all the equations is presented elsewhere (Wang and Melbourne 2010).

Table 1 : Comparison of live load effects

Pippard's equations	New equations
$$H = \dfrac{\displaystyle\int_A^B M_S y \dfrac{ds}{EI}}{\displaystyle\int_A^B y^2 \dfrac{ds}{EI}}$$	$$H = \dfrac{\displaystyle\int_A^B M_S y \dfrac{ds}{EI}}{\displaystyle\int_A^B y^2 \dfrac{ds}{EI} + \displaystyle\int_A^B \cos^2\alpha \dfrac{ds}{EA}}$$
Horizontal thrust at the crown due to a unit live load at the crown	
$$H_L = \dfrac{25}{128}\dfrac{L}{a}$$	$$H_L^T = \dfrac{1}{1+\lambda}H_L$$
Bending moment at the crown due to a unit live load at the crown	
$$M_L = \dfrac{7}{128}L$$	$$M_L^T = \dfrac{1+\dfrac{32}{7}\lambda}{1+\lambda}M_L$$
Stress at the crown extrados due to a unit live load at the crown (effective width of 2*h*)	
$$\dfrac{H_L}{2hd}+\dfrac{3M_L}{hd^2}$$ $$=\dfrac{L}{256hd}\left(\dfrac{25}{a}+\dfrac{42}{d}\right)$$	$$\dfrac{H_L^T}{2hd}+\dfrac{3M_L^T}{hd^2}$$ $$=\dfrac{1}{1+\lambda}\dfrac{L}{256hd}\left[\dfrac{25}{a}+\dfrac{42}{d}\left(1+\dfrac{32}{7}\lambda\right)\right]$$

where a is arch central rise, d is ring thickness at the crown, h is fill depth at the crown, L is arch span, and ρ is density of the fill and masonry (assumed to be the same).

Table 2 : Comparison of dead load effects

Pippard's equations	New equations
Horizontal thrust at the crown of a unit width bridge	
$$H_D = \dfrac{\rho L^2}{2a}\left(\dfrac{h+d}{4}+\dfrac{a}{21}\right)$$	$$H_D^T = \dfrac{\rho L^2}{2a}\left(\dfrac{h+d}{4}+\dfrac{a}{21}\right)\cdot\dfrac{1}{1+\lambda}$$
Bending moment at the crown of a unit width bridge	
$$M_D = -\dfrac{\rho L^2 a}{336}$$	$$M_D^T = -\dfrac{\rho L^2 a}{336}\left[\dfrac{1-7\lambda}{1+\lambda}-\dfrac{42(h+d)}{a}\cdot\dfrac{\lambda}{1+\lambda}\right]$$
Stress at the crown extrados due to the dead load	
$$\dfrac{H_D}{d}+\dfrac{6M_D}{d^2}$$ $$=\dfrac{\rho L^2}{2d}\left(\dfrac{1}{21}+\dfrac{h+d}{4a}-\dfrac{a}{28d}\right)$$	$$\dfrac{H_D^T}{d}+\dfrac{6M_D^T}{d^2}$$ $$=\dfrac{\rho L^2}{2d}\dfrac{1}{1+\lambda}\left\{\dfrac{1}{21}+\dfrac{h+d}{4a}-\dfrac{a}{28d}\left[(1-7\lambda)-\dfrac{42(h+d)}{a}\lambda\right]\right\}$$

In accordance with Pippard, for a point load W at the crown the compressive stress under the combined dead and live load together should not exceed the maximum permitted value of the compressive stress f_c i.e.

$$W\left(\dfrac{H_L}{2hd}+\dfrac{3M_L}{hd^2}\right)+\left(\dfrac{H_D}{d}+\dfrac{6M_D}{d^2}\right)\le f_c \tag{11}$$

Therefore, the limiting value of the point load at the crown derived by Pippard would be given by:

$$W = \frac{\dfrac{256 f_c hd}{L} - 128\rho Lh\left(\dfrac{1}{21} + \dfrac{h+d}{4a} - \dfrac{a}{28d}\right)}{\left(\dfrac{25}{a} + \dfrac{42}{d}\right)} \tag{12}$$

Similar to Eq.(11), the new limiting value of the point load at the crown will be given by:

$$W^T\left(\frac{H_L^T}{2hd} + \frac{3M_L^T}{hd^2}\right) + \left(\frac{H_D^T}{d} + \frac{6M_D^T}{d^2}\right) \le f_c \tag{13}$$

Therefore, instead of the original formula by Pippard, i.e. Eq.(12), the new limiting value of the point load at the crown

$$W^T = \frac{\dfrac{256 f_c hd}{L} - 128\rho Lh\dfrac{1}{1+\lambda}\left\{\left(\dfrac{1}{21} + \dfrac{h+d}{4a}\right) - \dfrac{a}{28d}\left[(1-7\lambda) - \dfrac{42(h+d)}{a}\lambda\right]\right\}}{\dfrac{1}{1+\lambda}\left[\dfrac{25}{a} + \dfrac{42}{d}\left(1 + \dfrac{32}{7}\lambda\right)\right]} \tag{14}$$

Considering that the two wheel loads could exist side by side, which corresponds to an axle load for a vehicle of normal track width, the safe axle load is

$$W_A = 2W \text{ or } W_A^T = 2W^T \tag{15}$$

Comparing with the safe axle load W_A by Pippard (Pippard 1948), the new safe axle load W_A^T obtained is relatively smaller for short span arch bridges, as shown in Table 3. The influence will be dependent upon the span/rise ratio, the ring thickness and the depth of the crown cover.
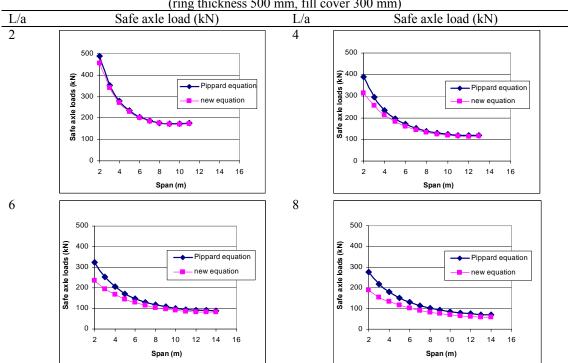
Table 3 Influence of λ on the safe axle load for different span rise ratio
(ring thickness 500 mm, fill cover 300 mm)

L/a	Safe axle load (kN)	L/a	Safe axle load (kN)
2		4	
6		8	

Table 3 demonstrate the influence of different span/rise ratio on the safe axle load for a bridge with a ring thickness of 500 mm and fill cover at the crown of 300 mm. For bridges with bigger ring thickness and fill cover, the influence of considering the axial thrust term will become more significant, i.e. much lower carrying capacity will be predicted for the relatively short span bridges.

3   OTHER ISSUES RELATED TO PIPPARD'S METHOD

3.1   Dead load stress

It should be pointed out that Pippard's original formula was based on limiting the compressive stress at the crown *extrados* under the *combined* dead and live load.

There are circumstances where either or both of the stress criteria assumed by Pippard i.e. f_c = 1400 kN/m^2 and f_t = -700 kN/m^2 (negative sign represents tensile stress) could be reached at the crown section under the self-weight only, as shown in Fig.3
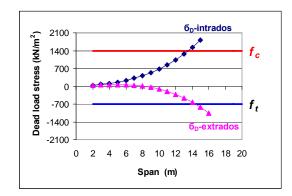

Figure 3 : Stress at the crown due to self weight (d=500 mm, h=300 mm)

It is important to note that Pippard did not consider this case even when his own stress limits were exceeded – this will be referred to elsewhere in the paper as 'overstress'.

Because Pippard did not considered this 'overstress' under the self-weight only, the safe axle load predicted using his original formula, would increase for higher spans (upturn of the curve for span over 13.5 m for the particular example considered), as shown in Fig.4.
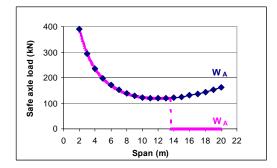

Figure 4 : Safe axle load using Pippard's original formula (d=500 mm, h=300 mm)

If, however, the 'overstress' under the self-weight is restricted, then the safe axle load will drop to zero for a bridge with a span over 13.5 m in this particular example, as also shown in Fig.4, due to the reason that the compressive stress at the crown intrados reaches f_c = 1400 kN/m^2 under the self-weight only.

The 'overstress' under the self-weight is restricted in the MEXE equations adopted by Network Rail Guidance Note (Network Rail 2006). Therefore, the curves for the safe axle loads from Network Rail version of MEXE are terminated suddenly for relatively larger span bridges.

3.2   Effective span

There is some uncertainty regarding the effective span to which Pippard applied his equations and their subsequent incorporation into the MEXE method. The method uses the clear span and rise, that is, the geometry of the intrados (Heyman 1982), whilst it might be more logical to use the neutral axis of the barrel. This would result in a larger span than the clear span between the face of the abutments (or piers). For large spans this will represent a small percentage error. However, in the case of small spans this will not be the case and much larger errors will be incurred. This is the subject of current research and will be reported in a future paper.

3.3   Tensile stress limit

It should be pointed out that the safe axle loads presented in Table 3 are based on limiting the compressive stress at the crown extrados less than $f_c = 1400$ kN/m^2. For some combination of ring thickness and crown cover, this compressive stress limit can be less restrictive compared with the tensile stress limit, i.e. $f_t = -700$ kN/m^2 and the carrying capacity could be controlled by the tensile stress at the crown intrados, as demonstrated in Fig.5.
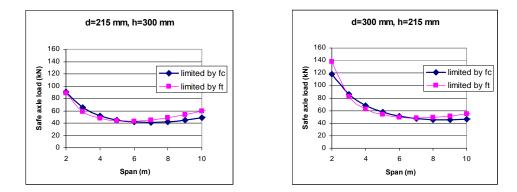

Figure 5 : Safe axle loads using Pippard's original equations (d+h=515 mm)

4   CONCLUSIONS

(1) Pippard neglected the effects of axial thrust in working out the strain energy. The error introduced is very small for most cases. However, it has been shown that for relatively small span thick arches (especially those with large span to rise ratios), the error becomes significant;

(2) New equations for determining the load carrying capacity of an arch that incorporate the effects of axial strain energy are presented, with which lower carrying capacities have been predicted for relatively small span bridges;

(3) Pippard's original work was based on limiting the compressive stress at the crown extrados under combined dead and live load. It has been shown that the stress criterion can be exceeded under the dead load only case for larger spans;

(4) Tensile stress limit could be in control in some circumstance, therefore, the axle load obtained from Eq.(15) should be checked against tensile stress limit before the final safe axle load is determined;

(5) Whilst the highway version of MEXE method would give the same value for the safe axle load for a given (h+d), the original equations give very different safe axle loads depending upon the relative values of h and d.

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