

Application of limit analysis to stone arch bridges

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ABSTRACT: The mechanism method, based on the particularization of limit design to masonry structures proposed by Heyman, is certainly a very suitable procedure for the analysis of stone arches, especially because of its simplicity, clarity and speed. In order to overpass the most important limit, which is the assumption of infinite compression strength, a simple procedure based on the reduction of the design thickness of masonry is proposed. The application of the mechanism method to multi-span arch bridges is also discussed.

1 INTRODUCTION

Limit analysis is certainly a very suitable method for the structural analysis of stone arch bridges. It is based on the assumption that masonry arches fail by forming pin joints, as demonstrated by old but also by more recent experimental studies (Clemente et al., 1999); as a result, the collapse must be viewed as a geometrical issue rather than a problem of strength of material. Heyman proposed the well known model of arch made up of a set of rigid voussoirs laid dry without any mortar and specified the limit design analysis to masonry.

The mechanism method presents some very important advantages (Gilbert, 2007). Among these: i) the simplicity, which means the method can be used in practical cases by any technician; ii) the clarity of the solution, which implies the solution can be checked and easily judged by the technician; iii) the speed, which allowed the analysis to be performed in not expensive time. Nowadays non-linear finite element computer codes are available, but these often appear too sophisticated and not always reliable for technical applications. On the other hand, some aspects not very clear by using failure analysis are to be investigated in more details. Two of these aspects are discussed in the following: the check of arches with not infinite strength in compression and the limit behaviour of multi-span bridges.

It is worth reminding that the hypothesis of infinite strength is certainly not suitable in reinforced masonry arches, where pin joints are not allowed at the edges and compression stresses can be high enough to cause crushing in masonry (Buffarini et al., 2006, 2007). On this aspect, an interesting method to improve failure analysis was proposed by Harvey who defined the thrust zone, which at each cross-section is of sufficient depth to carry the load, based on consideration of the actual material crushing strength. In this paper the reduction of the effective design thickness based on the material strength is proposed as a simple and suitable method for the limit check of stone arch bridges. Use of the mechanism method is also discussed with reference to multi-span bridges, by referring to a simplified model.

It is important to point out that this paper is devoted to long span bridges and to the possibility to propose new stone arches especially in historical cities. In these cases, the backfill is often substituted by means of a secondary structure, made of shorter arches, whose influence on the structural behaviour could be very important. Anyway this is out of the scope of this paper.

2 LIMIT ANALYSIS: REVIEW

The limit analysis of stone arches was developed by Heyman, who applied the ideas of plastic theory to masonry. He proposed the model of arch made up of voussoirs laid dry, for which the assumption that stone has no tensile strength is almost exactly true. In fact, although stone itself

may have some tensile strength, the joints will not, therefore no tensile forces can be transmitted from one voussoir to another. Besides, stresses are low enough to not allow crushing of the material. This observation is equivalent to the assumption that stone has an infinite compressive strength. He also assumed that friction between voussoirs is high enough to suppose that sliding failure cannot occur. With this assumptions the limit behaviour of a rectangular cross-section is completely described by the limit domain in the plane $e-N$ (e = eccentricity, N = axial force, t = depth), made of two horizontal straight lines:

$$\frac{e}{t} = \pm \frac{1}{2} \quad (1)$$

Failure of the arch occurs when sufficient hinges form to turn the structure into a mechanism. The arch is on the point of collapse if an equilibrium solution can be found, in which the resultant forces between voussoirs are always within the masonry and are at the arch profile at sufficient hinges to transform the structure into a mechanism. The uniqueness theorem ensure that this solution exists and is unique and so is the load factor. The safe theorem states that a structure is safe if an equilibrium solution can be found in which the resultant forces between voussoirs are always within the masonry.

Heyman's ideas were applied to study the limit behaviour and to find out the collapse mechanisms of stone arches under dead - plus - vertical live loads and to investigate the influence of the various parameters on the collapse mechanism and on the structural safety (Clemente et al. 1995, Clemente & Raithel 1998). It is important to remind that the thrust line of the safe theorem need not be the actual thrust line: every thrust line in equilibrium with external loads, lying within the arch profile, if any, can be chosen to check the structure. Moreover we do not know the actual stresses in the structure. In fact, we did not make any assumption about the material constitutive relationship, but the fact that the thrust line lies within the masonry ensures that there are only compressive actions, which can be transmitted from each voussoir to the next.

For any span L , the geometrical, mechanical and loading characteristics of the arch are individualized by the following non-dimensional parameters (Clemente & Raithel, 2001):

- (1) the sag ratio f/L ,
- (2) the thickness ratio t/L , where t can vary along the span,
- (3) the width b of the deck, usually assumed unitary,
- (4) the ratio $fd/(\gamma_w L)$, which define the material properties and tends to infinite according to Heyman's hypothesis,
- (5) the ratio between the weight per unit volume of the backfill and the weight per unit volume of the arch $\gamma = \gamma_b/\gamma_w$.

In the following simplified analysis the contributions of the fill and of the spandrel walls are ignored. The arch is composed by n voussoirs of length Δs . These can model the actual voussoirs or can just represent a geometrical discretization of the arch.

The dead load force for each voussoirs W_i can be expressed in the following ratios, independent of the span L (α_i is the average slope of the arch centre line for voussoir i):

$$\frac{W_i}{L^2} = \gamma_w \cdot b \cdot \left[\frac{t_i}{L} + \frac{\gamma_b}{\gamma_w} \cos \alpha_i \left(\frac{f}{L} + \frac{h}{L} + \frac{1}{2} \frac{t}{L} - \frac{y_i}{L} - \frac{1}{2 \cos \alpha_i} \frac{t_i}{L} \right) \right] \cdot \frac{\Delta s}{L} \quad (2)$$

The travelling force for each voussoir can be expressed in the ratio form, independent of the span L :

$$\frac{P_i}{L^2} = \frac{p \cdot \Delta s \cdot \cos \alpha_i}{L^2} = \frac{\Delta s \cdot \cos \alpha_i}{L} \sum_1^n \frac{W_i}{L^2} \quad (3)$$

where p is the reference value of the travelling load, which we assumed to be:

$$p = \frac{1}{L} \sum_1^n W_i \quad (4)$$

Let us consider the voussoir arch in Fig.1 and suppose that an equilibrium solution under dead loads can be found, in which the resultant forces between the voussoirs lie wholly within the masonry. When the travelling loads are put in action and are increased from zero to the collapse value the line of thrust changes and at least four hinges form.

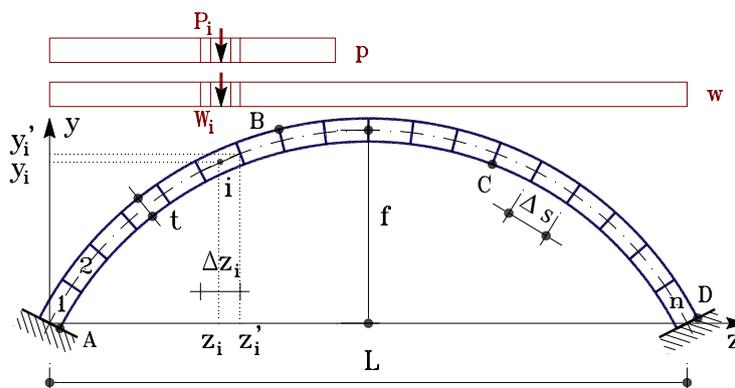


Figure 1 : The model in the mechanism method

On the point of collapse the resultant forces are at the hinge points in the hypothesised section and if the hinge is at an internal cross-section it must be tangential to the arch profile. The equation of equilibrium is given by the principle of virtual works:

$$L_w + \lambda L_p = 0 \tag{5}$$

in which (η_i = amplitude of virtual displacement diagram at the centre of voussoir i -th in the collapse mechanism):

$$L_w = \sum_1^n W_i \cdot \eta_i \quad L_p = \sum_1^n P_i \cdot \eta_i \tag{6}$$

It is very important to observe that in the equilibrium equation there are no terms with the internal forces acting at the hinges, whose values are still unknown. Consequently the equilibrium on the point of collapse is guaranteed by L_w which must be negative: dead loads have a stabilizing effect. Being the actual collapse mechanism not known, the solution will be found by using an iteration procedure that can be started giving a first mechanism and the corresponding diagram of virtual displacements.

For each load factor, associated to the given mechanism, we can find out the external reactions in the non-dimensional form: by means of the equilibrium equations:

$$\frac{H_A}{L^2} = \frac{H_p}{L^2} \quad \frac{V_A}{L^2} = \frac{V_D}{L^2} \tag{7}$$

and finally the axial forces, shear forces and bending moment, and then the eccentricity independent of the span:

$$\frac{N_i}{L^2} = \frac{T_i}{L^2} = \frac{M_i}{L^2} = \frac{e}{L} = \frac{M_i}{L^3} \cdot \frac{L^2}{N_i} \tag{8}$$

As already said the solution is found by using an iteration procedure. So if $e/t > 0.5$ at least at one section, a next step is needed in which we must move the hinges to the cross-sections where the ratio $|e/t|$ is maximum.

Considering only one uniform travelling load p acting from the centre-point of the left springing to the section at z_p co-ordinate, which simulates a load coming on to the structure a large numerical investigation was carried out (Clemente et al., 1995). The main results can be summarised as follows:

(1) the worst load conditions was first found, which corresponds to z_p a bit shorter than $L/2$, which is the classical half-span load condition. The small difference is due to the offset of hinges from the centre line, which results in a non perfectly anti-symmetric mechanism of 2. collapse with respect to the mid-span;

(2) when z_p/L tends to unity there is no danger of failure, because the hypothesized mechanism cannot be exploited under such a load condition;

(3) λ increases quickly as f/L decreases and increases with t/L .

3 THE LIMIT DOMAIN

Different constitutive relationship have been proposed for masonry, on which the limit domain is based. For our scope it is important the find out the resultant N and its position, i.e. the eccentricity e , or the bending moment $M = N \cdot e$, for each collapse point. In fact, we can always substitute the actual stress distribution with an equivalent uniform stress diagram of amplitude $f'_u = N/(b \cdot 2d)$, acting on a depth $2d$, $d = t/2 - e$ being the distance of N from the compressed edge. Stress f'_u is related to the maximum stress value of the effective constitutive relationship by means of a factor α , which depends on the shape and the effective depth of the diagram:

$$f'_u = \alpha \cdot f_u \tag{9}$$

Taking into account the material factor γ_m , we finally deduce the equivalent design compression strength, uniformly distributed on a depth $2d$:

$$f_d = \alpha \cdot f_u / \gamma_m \tag{10}$$

Usually it is $\alpha \approx 0.5$. The yield surface of a rectangular cross-section in the plane (N, M) is formed by two parabolic arcs (Fig.3, dotted line):

$$\frac{M}{N_u t} = \frac{1}{2} \frac{N}{N_u} \left(1 - \frac{N}{N_u}\right) \tag{11}$$

where $N_u = b \cdot t \cdot f_d$. The limit domain can also be written in terms of eccentricity (Fig.3, continuous line)

$$\frac{e}{t} = \frac{1}{2} \left(1 - \frac{N}{N_u}\right) \tag{12}$$

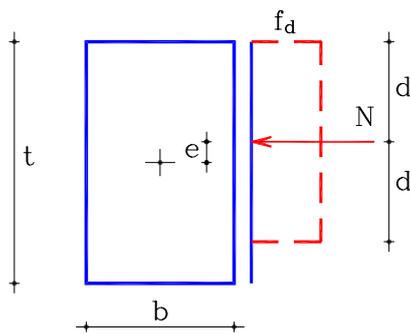


Figure 2 : Equivalent design stress distribution

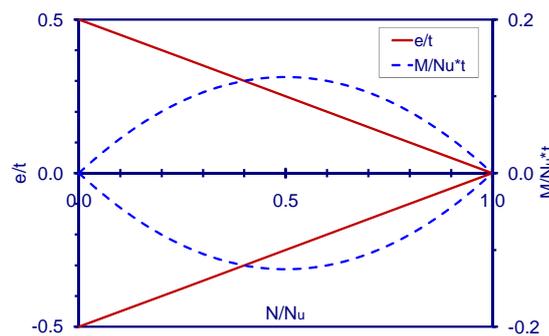


Figure 3 : Limit domain

4 THE DESIGN THICKNESS

In order to discuss some features of the analysis of masonry arches, let us consider the arch with the following parameters:

$$f/L = 0.25 \quad t/L = 0.25 \quad \gamma = 0.5 \quad z_p/L = 0.25 \quad b = 1 \quad (13)$$

In the following investigation the arch thickness t is assumed to be constant and a high number of voussoirs has been considered. So the obtained load factors are certainly not less than the actual one, which is related to the real size of the voussoirs. We introduce the non-dimensional finite compression strength of masonry, by means of the ratio:

$$f_d / (\gamma_w L) = 5 \quad (14)$$

which is a medium value in the practical range of interest. As a result it is:

$$N_u = f_d \cdot b \cdot t = 5 \cdot \gamma_w L \cdot b \cdot t \quad (15)$$

The classical limit analysis, carried out according to the Heyman's hypothesis of infinite compression strength and so with $t_d = t$, pointed out on the point of collapse the axial force distribution in Fig.4 (line with circles). As one can see, the eccentricity is equal to the limit value $e/t = 0.5$ at four sections, in which the hinges formed. These are called 1, 2, 3 and 4 in the following, instead of A, B, C and D, respectively.

By comparing the diagram $e-N$ relative to the limit analysis ($t_d = t$) with the limit domain we note that there are four zones, in correspondence of the four hinges (Hinges 1, 2, 3 and 4, from left to right along the arch – see Fig.1), in which the couples $e-N$ exceed the limit values. In order to bring the values inside the domain we can reduce the design thickness. In Fig.5 also two other $e-N$ distributions are plotted. The first ($t_d = t_2$) is relative to the situation in which only one hinge is statically admissible (Hinge 2, at the extrados) while at all the others the couples $e-N$ are out of the domain. In the second case the couple $e-N$ is on the limit domain only at one hinge (Hinge 1, at left springing); it is $t_d = t_1$ and the couples $e-N$ are in the domain at all the other sections.

In Fig.5 the same diagrams are plotted in the $M-N$ plane. This time the values $M-N$ at the hinges vary with linear laws. This suggests finding out the intersection points with the limit domain just analyzing two cases and then plotting the straight lines.

In Fig.6 the load factor is plotted against the effective thickness. The load factor λ_0 at $t_d = t$, is relative to the original mechanism method, in which the compression strength is assumed to be infinite. Obviously the load factor decreases when the thickness is lowered. The two points relative to the two values of the thickness t_1 and t_2 are pointed out. Actually, in none of them the arch is on the point of collapse, but in the first one is still safe (the hinge has been formed only at 2), in the second one is beyond the collapse point (the stress distributions are not compatible with the assumed material strength at 2, 3 and 4). So none of them represent a limit condition.

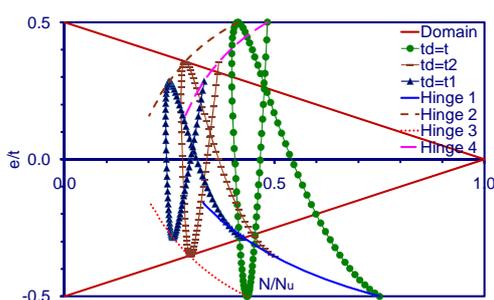


Figure 4 : Limit domain and axial force distribution in the e-N plane

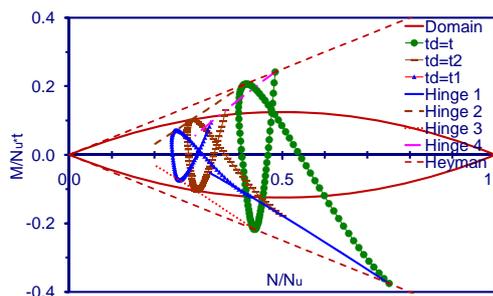


Figure 5 : Limit domain and axial Force distribution in the M-N plane

Anyway, from a practical point of view, we can say that:

- (1) if the lower load factor λ_1 , which is relative to t_1 , is high enough for the check of the arch, the analysis can be stopped;
 - (2) if the higher load factor λ_2 , which is relative to t_2 , is lower than the needed value, then the arch is certainly unsafe.
- if λ_2 is high enough but λ_1 is lower than the acceptable value, then a more detailed analysis is needed.

In the last case, and in general if the range of the thickness is too large, a more detailed analysis is needed in order to find out the limit condition in which the hinges will form contemporary at the four sections. Obviously, a variable thickness must be considered.

For the considered non-dimensional compression strength of masonry, $f_d/(\gamma_w L) = 5$, the load factor λ_1 is much lower than λ_0 , demonstrating the importance of a more detailed analysis about the material characteristics. It is important pointing out that the hinge locations are almost independent of the thickness as shown in Fig.7, where the non-dimensional curvilinear abscissa s/s_a (s_a = length of the arch axis) is plotted against the design thickness t_d/t . This is an important point in the numerical investigation. The influence of the thickness become important when $t_d < t/3$.

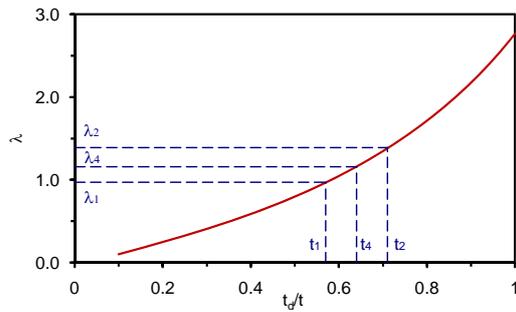


Figure 6 : Load factor against the design thickness

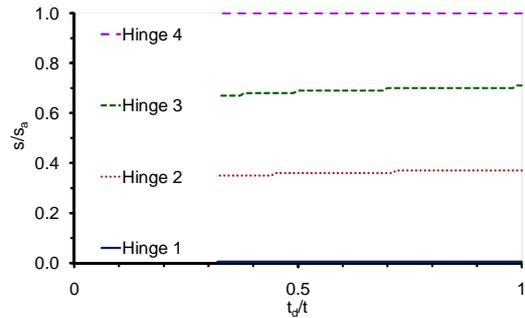


Figure 7 : Position of the hinges

5 MULTI-SPAN ARCH BRIDGES

Consider the system composed by two arches of same L, f and t (Fig.8). For simplicity refer to the original failure analysis, in which hinges are at intrados and extrados. The arch on the left is loaded by dead loads and travelling load acting on its left half span; the arch on the right is subject to dead loads only.

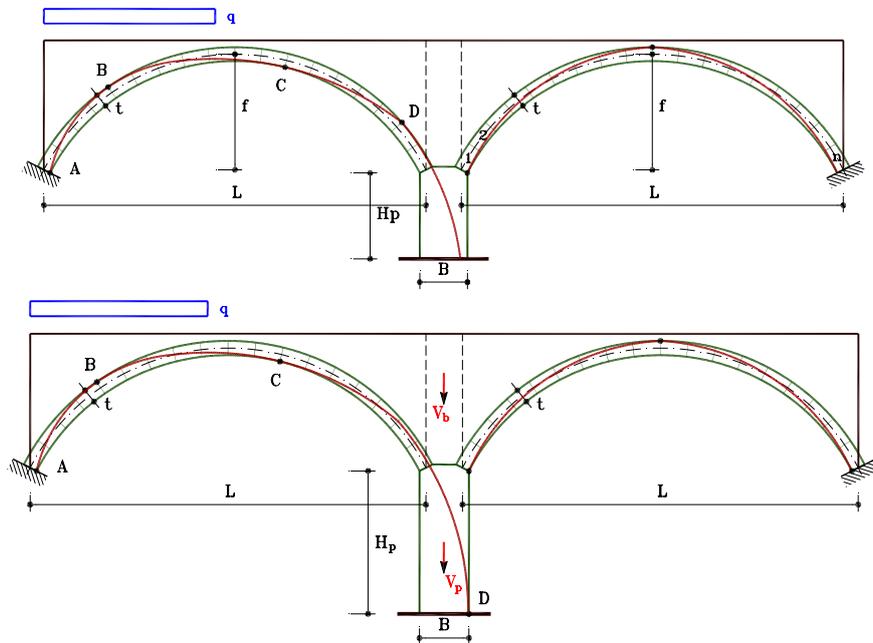


Figure 8 : Strong (upper) and weak (lower) pier \

Obviously the thrust line in the right arch is not defined; so, in order to be on the safe side, we suppose it passes at the extrados at the crown and at the intrados at springing. We distinguish two

kinds of behaviour: strong pier, in which the arch collapses while the pier is still safe; weak pier, when the arch-pier system collapses. The actual behaviour depends on the size of the pier. We define two limit values of the pier size: B_{min} and B_{max} .

When $B = B_{min}$, then the resultant force is at A' (Fig.9). If $B \leq B_{min}$ then the collapse load factor of the arch-pier system is certainly lower than the load factor of the arch only. The pier behaves as a weak pier.

If $B \geq B_{max}$ then the collapse mechanism cannot interest the pier and the two arches can be considered as separated structures (Fig.10), i.e. the right arch has no influence on the collapse of the left arch. The pier is always strong.

In Fig.11 the non-dimensional values of B_{min}/L and B_{max}/L are plotted against f/L for a typical value of t/L . Analogously, in Fig.12 the non-dimensional values of B_{min}/L and B_{max}/L are plotted against t/L for a typical value of f/L . In both cases the two curves represent the boundaries of the plane region in which the behavior depends on the height of the pier.

If $B_{min} \leq B \leq B_{max}$ then the behaviour depends on the height of the pier. In Fig.13 the minimum value of the height of the pier H_p/L , for which the behaviour is strong is plotted against B/L , for different values of the sag ratio f/L and with reference to a typical value of the thickness ratio t/L . Analogously, in Fig.14 the minimum value of the height of the pier H_p/L , for which the behaviour is strong is plotted against B/L , for different values of the thickness ratio t/L and with reference to a typical value of the sag ratio f/L .

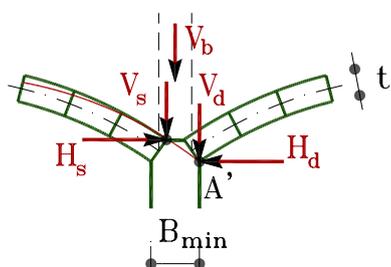


Figure 9 : $B=B_{min}$: weak pier

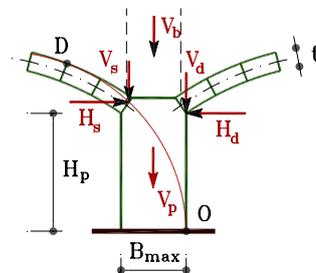


Figure 10 : $B=B_{max}$: strong pier

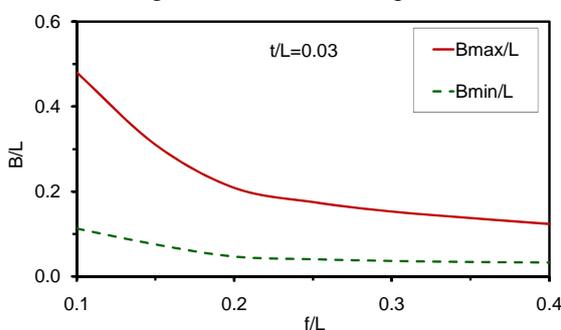


Figure 11 : B_{min}/L and B_{max}/L against f/L

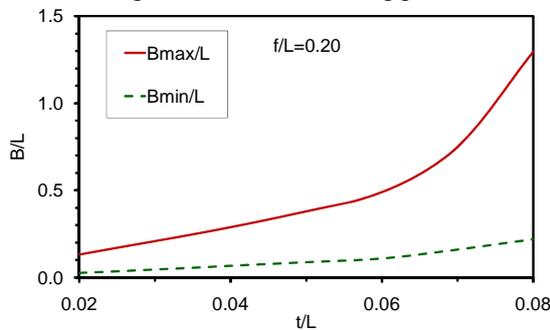


Figure 12 : B_{min}/L and B_{max}/L against t/L

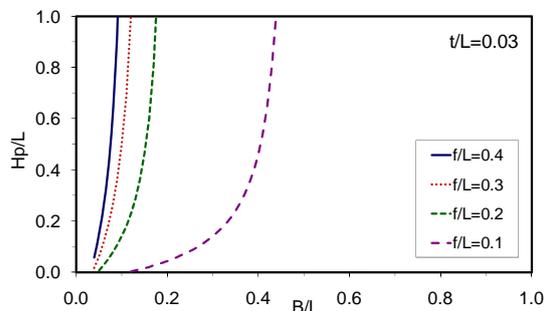


Figure 13 : $B_{min} \leq B \leq B_{max}$: H_p/L against B/L

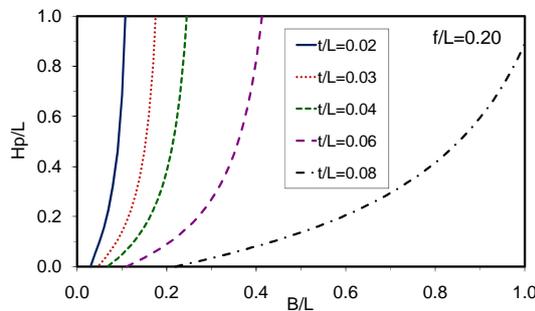


Figure 14 : $B_{min} \leq B \leq B_{max}$: H_p/L against B/L

6 CONCLUSIONS

The mechanism method still represents a very suitable procedure for the analysis of stone arch bridges. In fact, due to its simplicity, clarity and speed this method can easily be applied in most cases by technicians involved in the safety check of existing bridges. The most important limit, i.e. the assumption of infinite compression strength of the material, can be overpass by using the simple procedure here proposed, based on the reduction of the design thickness, related to the actual compression strength. The application to multi-span bridges is also possible by referring to very simple models.

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