ANALYSIS OF SOME HISTORICAL SIZING RULES FOR ARCHWALL-PIERS SYSTEMS

D. Aita, R. Barsotti, S. Bennati

University of Pisa, Department of Civil and Industrial Engineering, Pisa, ITALY.

e-mails: danila.aita@unipi.it, riccardo.barsotti@unipi.it, s.bennati@ing.unipi.it

SUMMARY

Starting in the 16th century, a number of scholars attempted to determine the minimum thickness of piers in arch-wall-piers systems so as to guarantee their stability. In this paper, we compare some of the many different sizing rules proposed during past centuries, and investigate the mechanical behaviour of the corresponding arch-wall-piers systems via modern limit analysis. In particular, an example application is studied: a system formed by an arch, an overhanging masonry wall and the piers bearing the arch and wall. The thickness of the piers is dimensioned according to some of the most significant historical sizing rules, and the mechanical behaviour of such systems examined via the Durand-Claye method, in order to assess their load capacity.

Keywords: *Masonry, arches, walls, piers, historical sizing rules, nonlinear elastic response, limit analysis, Durand-Claye method.*

1. INTRODUCTION

It is well known that the master builders of the Middle Ages possessed a body of knowledge that allowed them to design safe structures [1], despite the fact that they did not follow any scientific approach in the modern sense. In this context, the sizing of arch-piers systems is a subject of great interest. Medieval procedures to correctly design an arch-piers system seem to be deduced from observation of existing constructions. Starting in the 16th century some authors tried to codify such rules in order to determine the minimum thickness of piers so as to guarantee the stability of arch-wall-piers systems. The minimum thickness was obtained by means of geometric criteria, which varied from author to author and included: the intrados shape of the arch (*i.e.* Derand); the intrados shape and the arch thickness (*i.e.* Ruiz); the intrados shape, the arch thickness and the height of the superimposed wall and piers (*i.e.* Gil de Hontañón). Sizing rules based on mechanical principles emerged later, starting in the 17th century (*i.e.* De La Hire, Frézier, Danizy, Monasterio, Collignon).

The aim of this paper is not to provide an exhaustive account of these rules. A number of scholars have examined this subject by taking into consideration both the historical context and the contemporary building techniques. We refer the interested reader to the contributions by Heyman [2], Sanabria [3], Huerta [1, 4], Sakarovitch [5], Becchi and Foce [6], amongst others.

The main purpose of the present paper is instead to compare the effectiveness of such sizing rules from a mechanical point of view. More precisely, we consider the system formed by an arch, an overlying masonry wall and the piers bearing the arch and wall. The thickness of the piers is dimensioned according to a selected set of historical sizing rules. The mechanical behaviour of these systems is then analysed by means of the Durand-Claye method [7], as will be explained later.

2. HISTORICAL SIZING RULES

2.1. A brief overview

In this section we will review some historical sizing rules for arch-wall-piers systems. Some rules focus on the dimensioning of the piers; other rules take into account the overall system by also considering other parameters, such as the effect of the filling or the influence of the arch thickness.

The rule proposed by Martinez de Aranda (circa 1590) is a rather simple one that considers only the line of the arch intrados to determine the pier's thickness. This rule starts by dividing the intrados line into three equal parts. From the point thus obtained a straight vertical line is drawn which intersects the horizontal line defined by the arch's springings. The distance between this point and the intrados point at the springings provides the necessary pier thickness.

The rule set forth by de Aranda coincides - in terms of pier thickness - with that reported by Derand in his Treatise (1643), though the geometric description is different: divide the arc *abd* into three equal parts *ab*, *bc*, *cd*; starting from point *b,* trace the straight line *ba*. By imposing *ae* = *ab*, draw a straight line perpendicular to *fd*. The segment *af* will yield the required thickness. According to Huerta [1], this rule is very ancient. It can be found in various treatises, such as those by Blondel or De La Rue (see Fig. 1).

The rule proposed by Hernán Ruiz El Joven, described in his *Libro de Arquitectura*, 1560, considers both the intrados arc and the arch's thickness in order to size the pier's thickness. The procedure is as follows: draw the chord of the intrados semi-arc; trace a straight line parallel to that chord so that it is tangent to the extrados arc; the point of intersection between this tangent line and the horizontal straight line of the arch springings yields the pier thickness.

Sizing rule 7 by Gil de Hontañon (16th century) is particularly interesting because it correlates the arch thickness, the height of the overlying load, and the pier's height and thickness (for further details see [1] and [8]). For this reason, as we will show below, it will be chosen as the reference method for performing a comparison between various historical rules. This rule will be described in detail in paragraph 2.2.

The work of Philippe De La Hire, a professor at the Académie Royale d'Architecture in the seventeenth century, is of particular relevance to the matters at hand. Becchi [6] places De La Hire's contributions in the transition from *coupe des pierres* to *mécanique*. In his *Traité de la coupe des pierres*, however, no references can be found to confirm the passage from the former discipline to the latter. In the *Traité*, for example, the geometric Derand rule is reaffirmed. On the contrary, the rule proposed by De La Hire for an arch-wall-piers system (Fig. 2) testifies to the influence of stereotomy on the mechanical hypotheses adopted: the central *voussoirs* of an arch behave like "*une seule pierre*" [6, 9].

The theme of the monolithic arch is also present in other contributions on the mechanical behaviour of masonry arches. For example, Danizy's tests reported by Frézier [10] show that "*plus la clef est grande moins la poussée de la voute est grande*". Danizy excludes failure mechanisms by sliding and criticizes the mechanical hypotheses proposed by De La Hire.

Frézier's text (1737-1739) contains a number of sizing rules for arch-piers systems. Some are based on that proposed by De La Hire, who was Frézier's mentor.

For the sake of brevity, given their complexity, the rules by De La Hire, Frézier and Danizy will not be described in detail.

To conclude this cursory review, it is interesting to note in passing that Derand's geometric rule is once again reported in the stereotomy treatise by De La Rue (1764). In the 19th century some arch-piers system sizing rules relying on mechanical considerations were introduced; see for example the contributions of Monasterio and Collignon (1869).

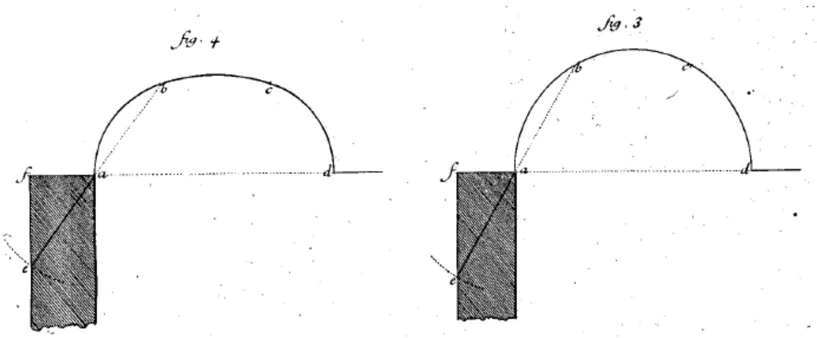

Fig. 1. *Derand's rule in the Treatise by De La Rue (1764).*

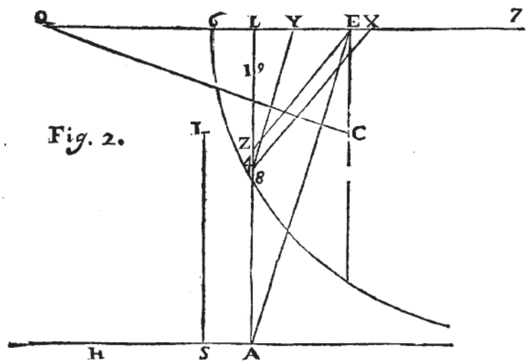

Fig. 2. *De La Hire's rule.*

2.2. An interesting rule by Gil de Hontañon

In his sizing rule 7, Gil de Hontañon considers a full-centre arch-pier system surmounted by a wall having a horizontal extrados (Fig. 3). The arch's intrados radius, R, and constant thickness, h, equal to 1/6 of the arch span, are known. Divide the span into three equal parts. The pier's thickness, B, is given by the distance EC, equal to 1/3 of the span. Consider the midpoint of the extrados crown section, A. Draw a straight line from this point to point C. The pier's height, H, will be defined by the intersection of this straight line with the vertical line passing through point E (segment ED). The maximum height of the overlying load, L, is defined by the point of intersection between the vertical line of the crown section and the circle with centre E and radius $2R + h$.

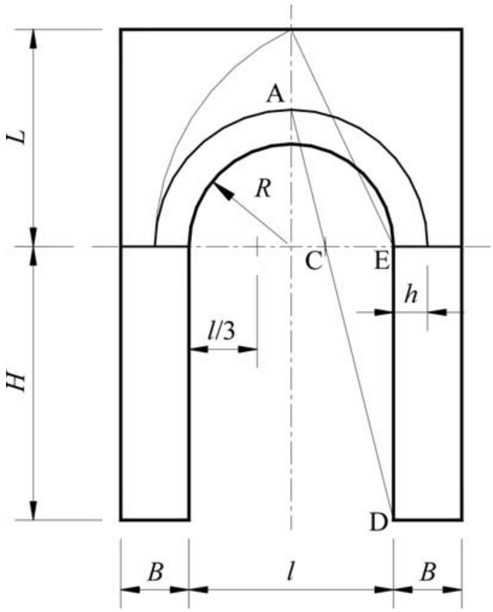

Fig. 3. *The sizing rule 7 by Gil de Hontañon.*

3. A FIRST COMPARISON BETWEEN THE SIZING RULES

The main objective of this work is to make a comparison between the most significant historical sizing rules for arch-wall-piers systems. In particular, the study attempts to clarify some issues on this subject, namely:

- the relation between geometric sizing rules and the mechanical modelling of arch-wall-piers systems;
- the field of application of each rule;
- the safety factor of each rule.

3.1. Determination of the pier's thickness

In order to compare the degree of safety yielded by these sizing rules, we start by considering an arch-wall-pier system dimensioned in accordance with Rodrigo Gil de Hontañon's rule 7, described in section 2.2.

Consider an arch-wall-piers system analogous to that drawn in Fig. 3. We set R = 5 m, while all the other dimensions are determined according to Gil's rule. In particular, we obtain: pier thickness, B = 3.33 m; intrados radius, R = 5 m; arch's thickness, h = 1.67 m; pier height, H = 13.33 m; overlying load height, L = 10.54 m.

With reference to the arch-wall-piers system described above, we now allow the pier's thickness, B, to vary, and determine its value according to the six design rules listed below (all the other geometric parameters are kept fixed). Thus, six arch-wall-piers systems have been defined:

1) De Aranda/Blondel/Derand (B = 2.50 m);
2) Gil de Hontañon (B = 3.33 m);
3) Danizy (B = 4.28 m);
4) Ruiz (B = 4.43 m);
5) De La Hire (B = 5.93 m);
6) Frézier (B = 6.69 m).

3.2. A study of the mechanical behaviour of the arch-wall-pier system via the Durand Claye method

In this section the mechanical behaviour and minimum pier thickness, B_{lim}, are investigated by using Durand-Claye's method, reworked through the Mathematica software package. For the sake of brevity, we do not describe the method in full detail here, and refer the interested reader to some previous works by the authors [11]. In short, the Durand Claye method enables determining all the admissible pairs of values (P, e), where P is the crown thrust and e its eccentricity at the crown section. The masonry is assumed to have limited compressive and tensile strength and a finite friction coefficient along any joint. The locus of the extremes of P in the (P, e) plane compatible with equilibrium and strength, called the area of stability, is determined by checking respect for the limitations on the strength at any given joint along the arch; when such set reduces to a single point, the arch reaches incipient collapse conditions.

In the following we assume that the arch, the piers and the superimposed wall have unit thickness in the transverse direction. The masonry has a unit weight γ = 20 kN/m^3, compressive strength, σ_c = 20 MPa, and a nil tensile strength; the friction coefficient is infinite. In this case, determining the pier thickness that reduces the area of stability to a single point (Figure 4, left) yields the limit pier thickness, B_{lim} = 2.54 m. Such limit thickness turns out to be slightly larger than that determined with the rule of De Aranda/Derand. The corresponding thrust line is plotted on the right of Figure 4. The incipient rotational collapse mechanism is represented on the left of figure 6. For more details, see for example [11] and [12].

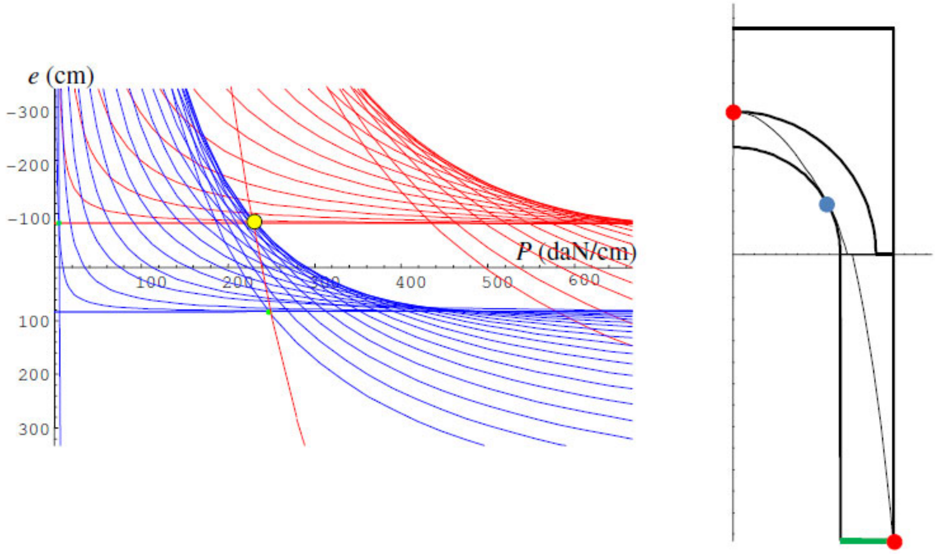

Fig. 4. Area of stability (left) and corresponding thrust line (right) for a limit thickness of the piers B_{lim} = 2.54 m.

In the following, the above-determined limit value for the pier thickness, B_{lim}, will be compared to those provided by the historical sizing rules considered. In this regard, it is worth observing that the Durand Claye method can be effectively used to check the mechanical interpretation of the rule proposed by De La Hire. De La Hire himself stated that his geometric rule, plotted in Fig. 2, corresponds to a mechanical scheme (that seems rather conventional at first sight) in which the arch is viewed as composed by three large *voussoirs*; frictionless contact conditions are moreover assumed between the central and two lateral parts.

In De La Hire's scheme only four joints are present. Thus, the Durand Claye method (Figure 5) is reduced to a check of the equilibrium of each *voussoir,* as well as of the limitations on the normal and tangential actions in correspondence to such joints (the same compressive and tensile strengths as before are assumed for the masonry). In this way, the minimum safe pier thickness, B_{DLH} = 4.93 m, is determined. In the corresponding collapse mechanism the central *voussoir* is lowered, while the two lateral parts rotate around the hinge at the pier's base (Fig. 6, right). Both mutual translation and rotation are observed between the *voussoirs*. It is worth noting that, given the monolithic nature assumed for the central *voussoir*, the point of application of the thrust at the crown section can be external to it (see the green point in Fig. 5). Lastly, note that the pier thickness determined according to De La Hire's geometric rule turns out to be slightly larger than that obtained by way of his mechanical scheme. In other terms, it seems that De La Hire included a margin of safety in his geometric rule.

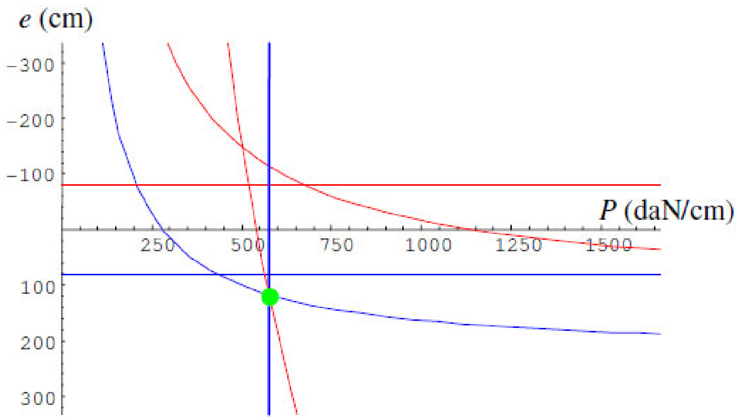

__Fig. 5.__ Area of stability for the limit thickness of the piers $B_{DLH} = 4.93$ m, assuming the mechanical model proposed by De La Hire.

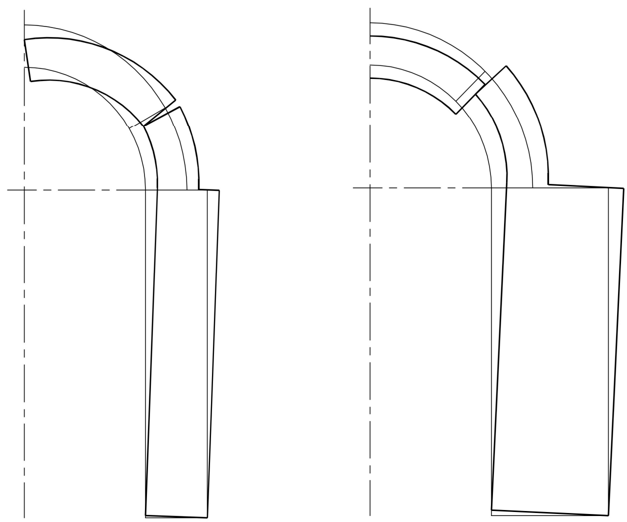

__Fig. 6.__ Rotational collapse mode (left) and mixed collapse mode "à la De La Hire" (right).

3.3. The degree of safety of the historical sizing rules

The six historical sizing rules examined in section 3.1 provide widely varying values of pier thickness, B. In order to assess their degree of safety, we define a "geometric" safety factor by taking the ratio $\beta = B/B_{lim}$.

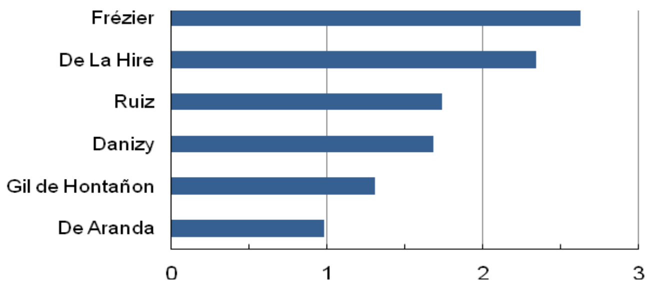

Fig. 7. *Geometric safety factor, β.*

By comparing the values of β (Fig. 7), it is interesting to observe that the geometric safety factors of the six rules, though very different, are all larger than unity, with the exception of De Aranda's (0.98). Thus, all the sizing rules seem to be able to provide safe values for the pier thickness. More precisely, the rules by Gil, Danizy and Ruiz yield safety factors of 1.31, 1.68, and 1.74, respectively.

It should be noted that the safety factors related to the De La Hire and Frézier rules (respectively 2.34 and 2.63) are considerably higher than the others. This fact can be explained by considering that these two authors based their sizing criteria on the mechanical model described in section 3.2, which makes use of two peculiar hypotheses, namely: the arch is made up of three large rigid blocks and the contact between the central and two lateral blocks is frictionless.

A second "mechanical" safety factor for piers can be defined as the maximum weight of the wall that the arch-piers system can withstand. To determine this, the shape and dimensions of the wall are kept fixed, while the unit weight γ of the wall is increased until the arch-wall-piers system reaches collapse conditions. The mechanical safety factor is defined as $\nu = \gamma_{max}/\gamma$, where $\gamma = 20$ kN/m^3 is the reference unit weight. The results are summarised in Table 1.

Table 1. *Geometric and mechanical safety factors.*

	Pier thickness, *B* (m)	**Geometric safety factor β**	**Mechanical safety factor ν**
De Aranda	2.50	0.98	0.88
Gil de Hontañon	3.33	1.31	7.19
Danizy	4.28	1.68	18.01
Ruiz	4.43	1.74	19.62
De La Hire	5.93	2.34	33.58
Frézier	6.69	2.63	34.00*

With reference to Table 1, the mechanical safety factor of the Frézier method, corresponding to a pier thickness of 6.69 m, has been evaluated by calculating the limit unit weight corresponding to collapse of the arch alone (since in this case incipient collapse conditions are not determined by the pier's height).

We limit ourselves to observing that very large mechanical safety factors are obtained as soon as the pier thickness reaches some 1.5 times the minimum value required for maintaining equilibrium in all parts of the arch-wall-piers system, B_{lim}. This finding confirms the well-known result that as long as the stresses are small with respect to masonry's compressive strength, the stability of a masonry system is ruled by its geometry. Further insights will be advanced in future contributions.

4. CONCLUSIONS

The study presented here is a first attempt at comparing the effectiveness of some historical sizing rules for arch-piers systems from a mechanical point of view. The rules considered provide very different results one from the other in terms of pier thickness. In order to investigate this issue, the degree of safety of applying such rules has been assessed by way of a theoretical study of the load capacity of the corresponding arch-wall-piers systems. The study, which utilises a suitably modified version of the Durand Claye method, has revealed that all these sizing rules seem to be quite safe, with the exception of De Aranda's. Larger pier thicknesses seem to result from application of the comparatively more recent rules by De La Hire and Frézier.

Many aspects still need to be clarified and therefore merit further study. In particular, it would be interesting to widen the survey to include other historical sizing rules and shed some light on the transition from dimensioning based on geometric and empirical criteria to a conception of sizing based on mechanical considerations. Another fundamental challenge is to understand what geometries or construction techniques correspond to each of the rules.

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