Limit analysis of masonry arches under vertical loads applied out middle plane

C. Anselmi, E. De Rosa and F. Galizia
Dipartimento di Costruzioni e Metodi Matematici in Architettura, Università Federico II, Naples, Italy

ABSTRACT: The objective of this work is to study the behaviour of masonry arches discretized in rigid voussoirs under vertical loads applied out of middle plane with the aim to evaluate the collapse multiplier of the above-mentioned loads. This evaluation is obtained through a procedure founded on the Static Theorem of Limit Analysis, with the assumption of limited compressive strength, inability to carry tension and sliding with dilatancy on the interfaces between the rigid blocks. This procedure has been already used with success by the authors for other typologies of masonry structures discretized in rigid blocks. In the present problem the computational process involves a limited number of unknowns and of conditions, and therefore the solution can be obtained easily through the Excel's solver. Once the load multiplier is obtained, the collapse mechanism is found.

1 INTRODUCTION

In this work we analyze the behavior of masonry arch-structures under vertical loads applied out of their middle plane through a very simple numerical method, already described in previous works and used with success by the authors for determining the collapse load multiplier of other typologies of masonry structures discretized in rigid blocks, such as panels and domes under increasing vertical or horizontal loads, and arches subjected to increasing loads applied in the middle plane. (Anselmi et al. 2004; Anselmi et al. 2006a, b; Anselmi et al. 2009a, b, c; De Rosa and Galizia 2007).

The masonry arch-structures are thought made of stones even dry assembled or connected with joints filled by mortar, modelled in rigid voussoirs.

With the computer program we have calculated the collapse multiplier of live load when the masonry arch is subjected booth to one force and to two forces applied on the extrados of the generic voussoirs, with variable eccentricity *e* from the middle plane of the arch.

The collapse load multiplier is evaluated utilizing the Static Theorem of Limit Analysis through a procedure of optimization constrained to respect of the equilibrium conditions and of the yield domains; we assume the following mechanical features on the contact interfaces:

(1) inability to carry tension;
(2) limited compressive strength;
(3) provision for blocks to slide with dilatancy.

The results obtained, with regard both to the values of the load multiplier and to the collapse mechanisms, appear encouraging because they seem "coherent".

2 THE PLANNING OF THE PROBLEM

2.1 The equilibrium conditions for the single voussoir

With reference to an arch with variable thickness, the generic voussoir (Fig. 1) is subjected in its centre of gravity to the own weight *P* and on the centroid of the generic interface to the six stress resultants - referred to the local reference frame *n*, *r* and *t* - named normal force N_n (or simply N), shear forces T_t and T_r, twist moment M_n, bending moments M_t and M_r, respectively applied on the interfaces *i* and *j* and expressed by:

$$\underset{(6\text{x}1)}{\mathbf{S}_i} = \begin{bmatrix} \underset{(3\text{x}1)}{\mathbf{R}_i} \\ --- \\ \underset{(3\text{x}1)}{\mathbf{M}_i} \end{bmatrix} = \begin{bmatrix} N_{ni} \\ T_{ti} \\ T_{ri} \\ --- \\ M_{ni} \\ M_{ti} \\ M_{ri} \end{bmatrix} \qquad \underset{(6\text{x}1)}{\mathbf{S}_j} = \begin{bmatrix} \underset{(3\text{x}1)}{\mathbf{R}_j} \\ --- \\ \underset{(3\text{x}1)}{\mathbf{M}_j} \end{bmatrix} = \begin{bmatrix} N_{nj} \\ T_{tj} \\ T_{rj} \\ --- \\ M_{nj} \\ M_{tj} \\ M_{rj} \end{bmatrix}$$

In Fig.1 we have denoted with b the width of the arch, while with s_i and s_j the thickness of the interfaces i and j respectively.

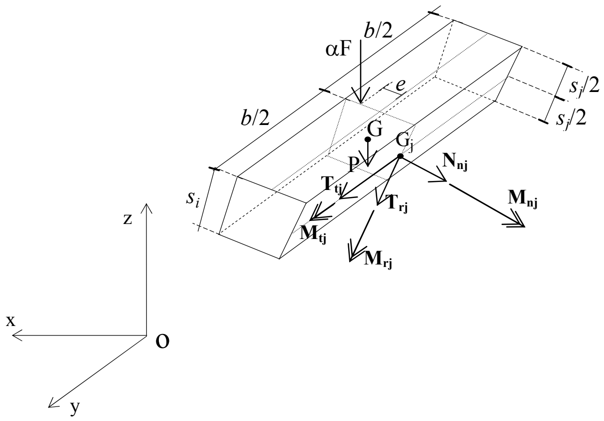


Figure 1 : Generic voussoir. Local and global reference frame. Stress resultants on the generic interface j.

The equilibrium conditions for the generic voussoir, in the global reference frame x, y and z, are so formulated:

$$\mathbf{A}^e\mathbf{X}^e + \mathbf{F}_0^e + \alpha\mathbf{F}_1^e = \mathbf{0} \tag{1}$$

where $\mathbf{A}^e$ is a matrix (6x12) of the coefficients of the unknown stress resultants $\mathbf{X}^e$ depending on lying and on the dimensions of the voussoir, $\mathbf{F_0}^e$ is the vector of the dead loads and $\mathbf{F_1}^e$ the vector of the live load - when is present on the voussoir - increasing by the multiplier α.

2.2 Yield domain for the generic interface

The stress resultants on the generic interface j have to respect in all eighteen yield conditions of the material and precisely twelve conditions for rocking domain (Fig.2a) and six conditions for sliding domain (Fig.2b) as showed later on.

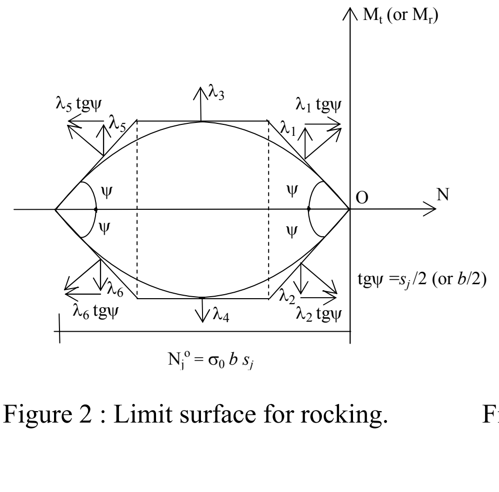


Figure 2 : Limit surface for rocking.

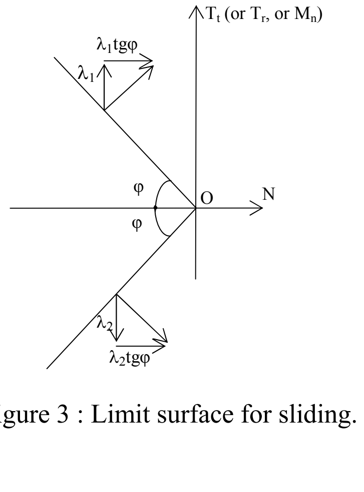


Figure 3 : Limit surface for sliding.

2.3 Yield domain normal force N - bending moment M

The closed parabolic N-M yield domain of Fig.2 has been opportunely replaced by a piecewise linear yield domain having six sides.

Therefore, with reference to the two bending moments, we impose six conditions for the yield domain ($N - M_t$):

$$\frac{s_j}{2} N \pm M_t \leq 0\,; \qquad \pm M_t \leq \frac{\sigma_0 s_j^2 b}{8}\,; \qquad -\frac{s_j}{2} N \pm M_t \leq \frac{\sigma_0 s_j^2 b}{2} \tag{2}$$

and six for the yield domain ($N - M_r$):

$$\frac{b}{2} N \pm M_r \leq 0\,; \qquad \pm M_r \leq \frac{\sigma_0 b^2 s_j}{8}\,; \qquad -\frac{b}{2} N \pm M_r \leq \frac{\sigma_0 b^2 s_j}{2} \tag{3}$$

being σ_0 the compressive stress limit.

2.4 Yield domain N – shear force T

This domain defined by a cone, with axis coinciding with the N-axis has been opportunely replaced by a piecewise linear yield domain having four sides. Therefore we impose four conditions making reference to the Cartesian components T_t and T_r of T:

$$\tan\varphi_0 N \pm T_t \leq 0 \tag{4}$$

$$\tan\varphi_0 N \pm T_r \leq 0 \tag{5}$$

Thus, in Fig.3, T coincides with the generic component T_t or T_r, whereas φ coincide with the angle of friction φ_0.

2.5 Yield domain N - twisting moment M_n

Denoting with $\tau° b^2 s_j /4$ the moment M_n of an couple of forces $T° = \tau° b s_j /2$ having arm $b/2$ (Fig.4), we impose the following two conditions:

$$\tan\varphi_0 \frac{b}{4} N \pm M_n \leq 0 \tag{6}$$

eing $\tau°$ the limit tangential stress, $N = \sigma b s_j$ and $\tau°/\sigma = \tan\varphi_0$ the yield condition between limit tangential stress $\tau°$ and normal stress σ in the domain defined by the cone with axis coinciding with the σ-axis.

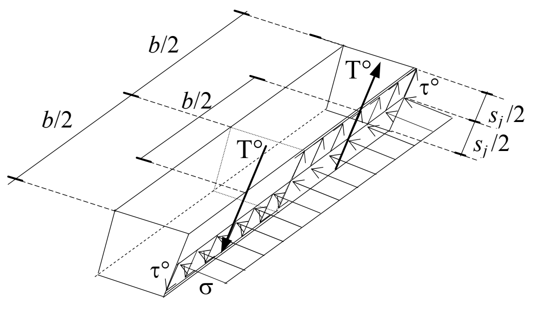


Figure 4 : Generic voussoir. Tangential limit stress $\tau°$ and normal stress σ on the generic interface *j*.

3 GOVERNING CONDITIONS

If *n* and *m* are respectively the number of the voussoirs and of the interfaces, the equilibrium conditions are:

$$\mathbf{A}\,\mathbf{X} + \mathbf{F_0} + \alpha\,\mathbf{F_1} = \mathbf{0} \tag{7}$$

and the yield domain's conditions are:

$$\mathbf{Y} = \mathbf{DX} - \mathbf{B} \leq \mathbf{0} \tag{8}$$

where **A** is a (6*n* x 6*m*) matrix, **X** is a 6*m* vector, $\mathbf{F_0}$ and $\mathbf{F_1}$ are 6*n* vectors, α is the unknown collapse multiplier, **D** is a (18*m* x 6*m*) matrix and **B** is a 18*m* vector of known terms.
The problem is resolved researching the maximum α subject to (7) and (8), with $\alpha \geq 0$.

4 THE EVALUATION OF COLLAPSE MECHANISM

Once the multiplier α has been calculated we can obtain the collapse mechanism having in account the conditions:

$$\mathbf{A}^{\mathrm{T}}\mathbf{u} = \boldsymbol{\Delta} \tag{9}$$

and of the flow rule:

$$\boldsymbol{\Delta} = \mathbf{D}^{\mathrm{T}}\boldsymbol{\lambda} \tag{10}$$

being **u** the vector of the degrees of freedom (six for every voussoir), $\boldsymbol{\Delta}$ the vector which collects the displacements between the interfaces (six for every interface) and $\boldsymbol{\lambda}$ the vector of the generalized strain rates associated to the yield conditions (*18* for every interface).
We have pursued both the kinematic and static solution making use of *Excel.*

5 APPLICATIONS

In order to evaluate the influence of live vertical load F applied out of middle plane of the arch, in this first approach we have studied an arch-bridge with semicircular profile having middle radius R=6m and springer angle β= 30°, discretized in eleven rigid voussoirs (Fig.5).

We have considered three different load conditions: in the first one, only one live load F is applied on a generic single voussoir; in the second and in the third condition two equal live loads F are applied respectively once on two generic near voussoirs but with a voussoir unloaded among of them, and once on two generic voussoirs symmetrically placed with respect to the keystone.

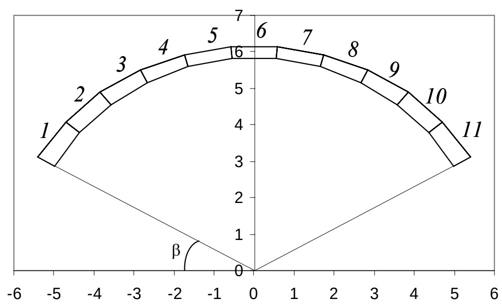


Figure 5: arch discretized in eleven voussoirs

The live loads lie always in vertical planes orthogonal to the middle plane of the arch, each of them passing for the centroid of the extrados face of the generic voussoir.

The live loads eccentricity has been expressed by $e=d(b/2)$, being *b* the width of the arch and *d* an adimensional parameter varying from zero to one. In all the applications it has been assumed $\tan\varphi_0 = 0.4$ and specific gravity γ=15.69 KN/m^3.

5.1 One live load F on a generic voussoir

We have considered one load F increasing through the multiplier α applied time to time on a generic voussoir, beginning from the voussoir 2 until the keystone (voussoir 6), and varying *d* from zero to one. The results obtained are reported in Fig.6. The Fig.7 shows instead the diagrams *alfa - voussoir loaded*, for fixed values of *d*.

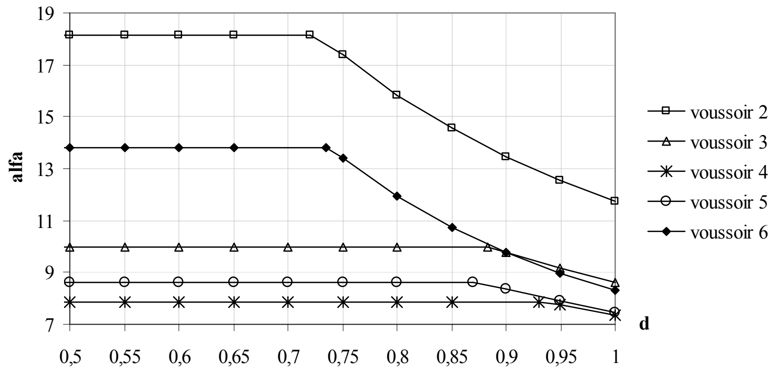


Figure 6 : α–d diagram, with one live load F on different voussoirs

By way of example the Fig.8 and 9 show respectively the collapse mechanisms for the arch subject to load F applied on the voussoirs 4 and 6, once with *d*=0 and once with *d*=1.

At last, in the case of the vertical load F applied on the keystone, we have evaluate the influence of friction coefficient tanφ0 - with values ranging from 0.2 to 0.7 - once with d=0 and once with d=1. The diagram (Fig.10) shows that the collapse multiplier increases when tanφ0 increases, until it stabilizes for values of tanφ0 such that do not arise sliding crisis but rocking crisis only.

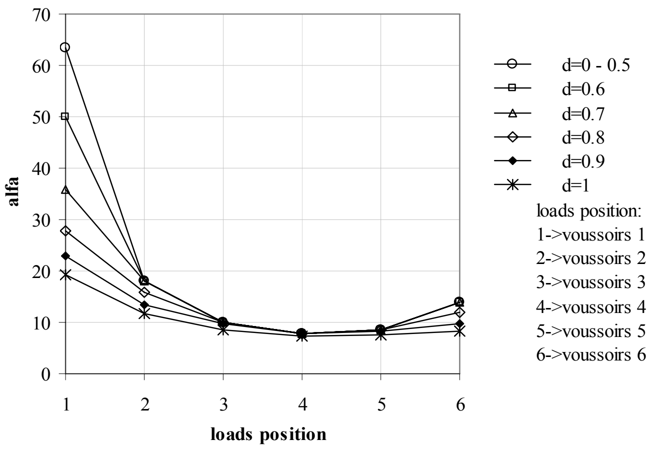
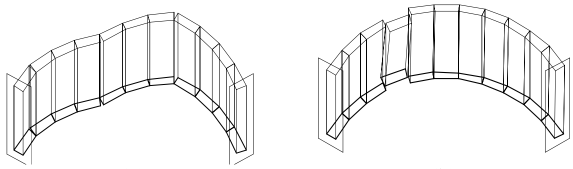


Figure 7 : $\alpha -$ *load position* diagram, for fixed values of *d*



(a) (b)

Figure 8: collapse mechanisms for the arch subject to load F applied on the voussoir 4. (a) *d*=0 , (b) *d*=1

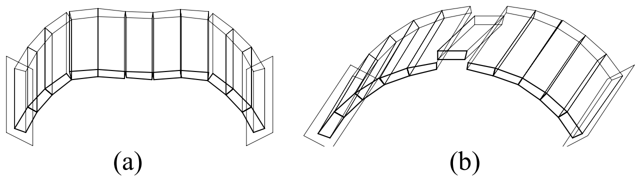


Figure 9: collapse mechanisms for the arch subject to load F applied on the voussoir 6. (a) d=0 , (b) d=1

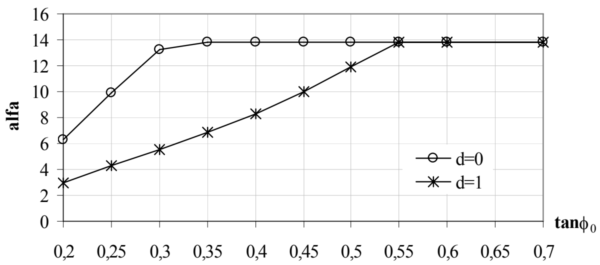


Figure 10: α–$tan\phi_0$ diagram with live load F on keystone, for *d*=0 and *d*=1

5.2 Two vertical loads F on two near voussoirs with a voussoir unloaded among of them

We have considered two equal live loads F applied on two near voussoirs but with a voussoir unloaded among of them, increasing through the multiplier α and with eccentricity *d* variable from zero to one. The Fig.11 shows, for every load condition beginning from the voussoirs 1-3 until the voussoirs 5-7, the diagram of α for *d*=0 and *d*=1.

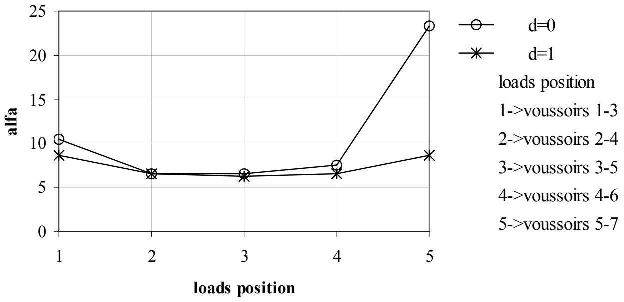


Figure 11: *load position* diagram with live load F on two near voussoirs with a voussoir unloaded among of them, for *d*=0 and *d*=1

By way of example the Fig.12 shows the collapse mechanisms for the arch subject to two loads F applied on the voussoirs 3-5, once with *d*=0 and once with *d*=1.

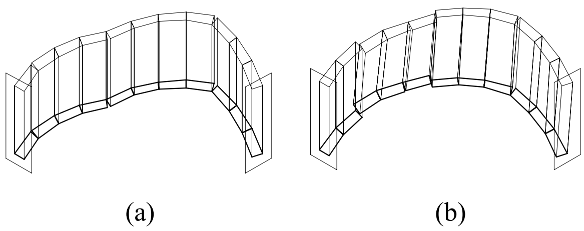


Figure 12: collapse mechanisms for the arch subject to load F applied on the voussoirs 3-5. (a) *d*=0 , (b) *d*=1

5.3 Two vertical loads F on two generic voussoirs symmetrically placed with respect to the keystone

We have considered two loads F applied on two voussoirs symmetrically placed with respect to the keystone, increasing through the multiplier α and with eccentricity *d* variable from zero to one. The Fig.13 shows, for every load condition beginning from the voussoirs 1-11 until the voussoirs 5-7, the diagram of α with *d*=1. The Fig.14 shows, for every load condition, the diagram of α with *d* increasing.

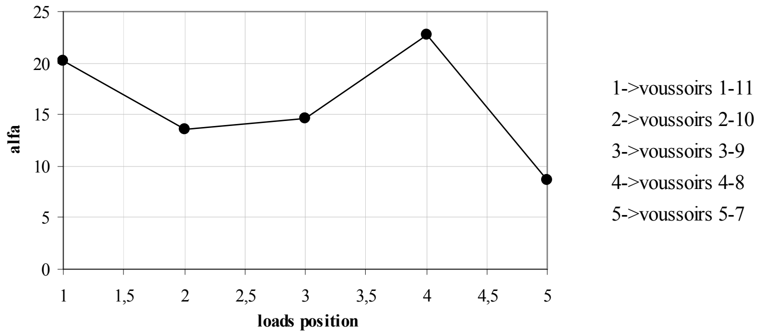


Figure 13 : $\alpha - load\ position$ diagram with live loads F on two voussoirs symmetrically placed with respect to the keystone, for *d*=1

By way of example the Fig.15 and 16 show the collapse mechanisms for the arch subject to two loads F with eccentricity *d*=0 and *d*=1 applied on the voussoirs 3-9 and 5-7 respectively.

Lastly, the Fig.17 shows the collapse mechanisms for the arch subject to two loads F on the voussoirs 4-8. In this particular case -unlike all previous cases examined in which the rocking crisis arise for small values of the compressive strength corresponding to the two straight lines near to the M-axis in Figure 2-, for d tending to zero, the compression strengths reach very high values and crisis for crushing occur (lines parallel to N-axis in Fig.2).

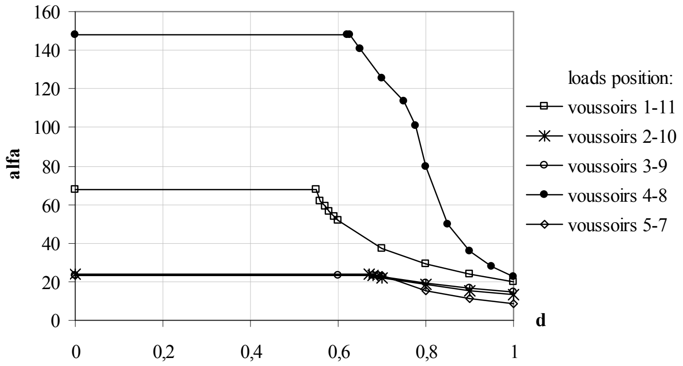
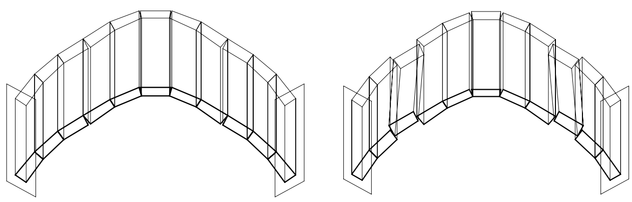
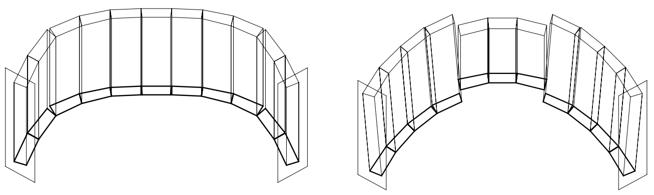


Figure 14 : $\alpha - d$ diagram with live loads F on two voussoirs symmetrically placed with respect to the keystone, for different loads positions.



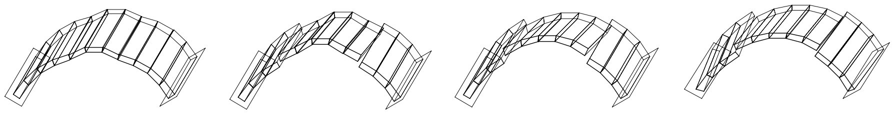
(a) (b)

Figure 15: collapse mechanisms for the arch subject to load F applied on the voussoirs 3-9. (a) *d*=0, (b) *d*=1



(a) (b)

Figure 16: collapse mechanisms for the arch subject to load F applied on the voussoirs 5-7. (a) *d*=0 , (b) *d*=1



(a) (b) (c) (d)

Figure 17 : collapse mechanisms for the arch subject to load F applied on the voussoirs 4-8.
(a) *d*=0-0.62 , (b) *d*=0.63-0.64 , (c) *d*=0.65-0.76 , (d) *d*=0.77-1

6 CONCLUSIONS

In this work we have evaluated the collapse multiplier of a masonry arch-bridge with semicircular profile having middle radius R=6m and springer angle β= 30° discretized in eleven rigid voussoirs under vertical live loads applied out of their middle plane. The collapse load multiplier has been obtained utilizing the static Theorem of Limit Analysis through a procedure of optimization constrained to respect of equilibrium conditions and of yield domains, under the following mechanical features: inability to carry tension for the contact interfaces, limited compressive strength at interfaces, provision for blocks to slide with dilatancy.

We have considered three different load conditions. In the first condition, only one live load F is applied on a generic single voussoir; in the second and in the third one two equal live loads F are applied respectively once on two generic near voussoirs but with a voussoir unloaded among of them, and once on two generic voussoirs symmetrically placed with respect to the keystone. We have also considered the influence of the angle of friction φ_0.

The computer program compiled involves a limited number of unknowns and both the load multiplier and the collapse mechanism has been obtained easily through the Excel's solver. Even though at this moment our applications are related always to the same arch, the program allows easily to change geometric parameters, voussoirs number and also to consider the fill and possible distributed vertical up-loads.

The results obtained confirm the expectations: when the eccentricity *e* tends to the edge of arch, the load multiplier decreases and a kinematic mechanism out of the middle plane arises.

Currently it has been possible to compare our results only with those obtained by Livesley (1992) and, as regards the qualitative aspects, the comparison seems to be very comforting.

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